3.48. Let Y denote the number of radar sets detecting the missile; i.e., $Y \sim b(n, p = 0.9)$.
(a) Let $n = 5$; i.e., $Y \sim b(n = 5, p = 0.9)$. We have

$$P(Y = 4) = \binom{5}{4}(0.9)^4(0.1)^1 \approx 0.328$$

and

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{5}{0}(0.9)^0(0.1)^5 = 1 - 0.00001 = 0.99999.$$

In R, these are calculated as

```
> dbinom(4,5,0.9)
[1] 0.32805
> 1-pbinom(0,5,0.9)
[1] 0.99999
```

(b) Suppose $Y \sim b(n, p = 0.9)$. The missile will be detected when at least one radar set detects the missile; i.e., when the event $\{Y \geq 1\}$ occurs. The probability of this event is

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{n}{0}(0.9)^0(0.1)^n = 1 - (0.1)^n.$$

We want this probability to be 0.999. Therefore,

$$1 - (0.1)^n = 0.999 \implies n = 3.$$

We would need $n = 3$ radar sets to have this reliability level.

3.54. If $Y \sim b(n, p)$, then Y counts the number of successes in n Bernoulli trials. The random variable $Y^* = n - Y$ therefore counts the number of failures.
(a) From what I can tell, part (a) is obvious. The event $\{n - Y = y^*\}$ and $\{Y = n - y^*\}$ are the same event; i.e., just rewrite using algebra. Therefore, they have the same probability.
(b) For $y^* = 0, 1, 2, ..., n$, we have from part (a),

$$\begin{aligned} P(Y^* = y^*) = P(Y = n - y^*) &= \binom{n}{n - y^*} p^{n-y^*}(1 - p)^{n-(n-y^*)} \\ &= \binom{n}{y^*}(1 - p)^{y^*} p^{n-y^*}. \end{aligned}$$

This shows that the number of failures $Y^* \sim b(n, 1 - p)$. Note that $\binom{n}{n-y^*} = \binom{n}{y^*}$ from Exercise 2.68 (HW2).
(c) I don't know that it is obvious, but it makes sense intuitively. Simply interchange the meaning of "success" and "failure."

3.62. (a) We would have to assume that the events

$$\begin{aligned} A &= \{\text{inspect plane that has a wing crack}\} \\ B &= \{\text{inspect detail where crack located}\} \\ C &= \{\text{detecting the damage}\} \end{aligned}$$

are mutually independent with $p_1 = P(A)$, $p_2 = P(B)$, and $p_3 = P(C)$. Detecting the crack would occur when $A \cap B \cap C$ occurs. Under the mutually independence assumption,

$$P(A \cap B \cap C) = P(A)P(B)P(C) = p_1 p_2 p_3.$$

(b) Let Y denote the number of planes where a wing crack is detected. Then $Y \sim b(n = 3, p = 0.36)$. Then

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{3}{0}(0.36)^0(0.64)^3 \approx 1 - 0.262 = 0.738.$$

In R,

```
> 1-pbinom(0,3,0.36)
[1] 0.737856
```

3.66. (a) Showing the geometric pmf sums to 1 was done in the notes; see pp 58.
(b) For $y = 1, 2, 3, ...,$ the pmf of $Y \sim$ geometric(p) is $p_Y(y) = q^{y-1}p$, where $q = 1-p$. Therefore, for $y = 2, 3, 4, ...,$

$$\frac{p_Y(y)}{p_Y(y-1)} = \frac{q^{y-1}p}{q^{(y-1)-1}p} = \frac{1}{q^{-1}} = q,$$

as claimed. Because $q < 1$, note that

$$p_Y(y) = qp_Y(y-1) < p_Y(y-1),$$

for $y = 2, 3, 4, ...$ In other words, $p_Y(1) > p_Y(2) > p_Y(3) > p_Y(4) > \cdots$. This means $y = 1$ is the most likely value in the geometric distribution; i.e., the mode of Y is $y = 1$.

3.71. (a) We are given that a is a positive integer. Using the complement rule, we have

$$P(Y > a) = 1 - P(Y \leq a) = 1 - \sum_{y=1}^{a} q^{y-1}p \overset{x=y-1}{=} 1 - p\sum_{x=0}^{a-1} q^x.$$

Note that

$$\sum_{x=0}^{a} q^x = \frac{1-q^a}{1-q}$$

because $\sum_{x=0}^{a-1} q^x$ is a finite geometric sum with common ratio q. Therefore, because $1 - q = p$, we have

$$P(Y > a) = 1 - p\left(\frac{1-q^a}{1-q}\right) = 1 - (1 - q^a) = q^a,$$

as claimed.
(b) Recall the definition of conditional probability and write

$$P(Y > a + b|Y > a) = \frac{P(Y > a + b \text{ and } Y > a)}{P(Y > a)} = \frac{P(Y > a + b)}{P(Y > a)}.$$

The last step is true because $\{Y > a+b\} \subset \{Y > a\}$ so that $\{Y > a+b\} \cap \{Y > a\} = \{Y > a+b\}$. Therefore,

$$P(Y > a + b|Y > a) = \frac{P(Y > a + b)}{P(Y > a)} = \frac{q^{a+b}}{q^a} = q^b.$$

This shows that

$$P(Y > a + b | Y > a) = P(Y > b)$$

which is a condition known as the memoryless condition. The geometric random variable is the only discrete random variable that satisfies this property.

Interpretation: Suppose Experimenter 1 is observing Bernoulli trials, and the first success has not occurred in the first a trials. This is what is meant by the "given" event $\{Y > a\}$. The probability she has to wait an additional b trials to observe the first success; i.e., $\{Y > a + b\}$ is the same as for another experimenter, say Experimenter 2, having to wait b trials from the outset. In other words, the fact that Experimenter 1 has not observed a success in the first a trials has been "forgotten."

(c) I don't know that this is obvious, but it is certainly true. I think the key is that we are waiting for the "first success." Because the trials are independent, observing a bunch of failures from the outset doesn't affect future trials and hence does not impact when we will observe the first success.

3.77. Suppose $Y \sim$ geometric(p). We want to calculate

$$\begin{aligned} P(Y = \text{odd integer}) &= P(Y = 1) + P(Y = 3) + P(Y = 5) + P(Y = 7) + \cdots \\ &= p + q^2p + q^4p + q^6p + \cdots \\ &= p\left(1 + q^2 + q^4 + q^6 + \cdots\right) \ = \ p\sum_{j=0}^{\infty}(q^2)^j. \end{aligned}$$

Note that $\sum_{j=0}^{\infty}(q^2)^j$ is an infinite geometric sum with common ratio q^2. Therefore,

$$\sum_{j=0}^{\infty}(q^2)^j = \frac{1}{1-q^2}.$$

The result follows immediately.

3.97. In this problem, we envision each oil well as a "trial," where "success" means that the well produces oil (i.e., "strikes" oil). Assume the oil wells are independent, each with probability of success $p = 0.2$. These are the assumptions needed for the question in part (c); i.e., the Bernoulli trial assumptions hold.

(a) Let Y denote the number of wells observed to find the first productive well (i.e., the first success). Then $Y \sim$ geometric($p = 0.2$) and

$$P(Y = 3) = (1 - 0.2)^2(0.2) = 0.128.$$

In R,

```
> dgeom(3-1,0.2)
[1] 0.128
```

(b) Let X denote the number of wells to find the third productive well (i.e., the third success). Then $X \sim \text{nib}(r = 3, p = 0.2)$ and

$$P(X = 7) = \binom{7-1}{3-1}(0.2)^3(1-0.2)^4 \approx 0.049.$$

In R,

```
> dnbinom(7-3,3,0.2)
[1] 0.049152
```

(c) See discussion above.
(d) In this part, we want $E(X)$ and $V(X)$ in part (b). With $r = 3$ and $p = 0.2$, we have

$$E(X) = \frac{r}{p} = \frac{3}{0.2} = 15 \text{ wells.}$$

Also,

$$V(X) = \frac{rq}{p^2} = \frac{3(0.8)}{0.2^2} = 60 \text{ (wells)}^2$$

3.159. In Exercise 3.158, Y is a random variable with mgf $m_Y(t)$. The random variable $W = aY + b$ is a linear function of Y. You showed in HW4 that $m_W(t) = e^{bt}m_Y(at)$. To find $E(W)$ note that

$$\frac{d}{dt}m_W(t) = \frac{d}{dt}e^{bt}m_Y(at) = be^{bt}m_Y(at) + e^{bt}m'_Y(at) \times a = e^{bt}\left[bm_Y(at) + am'_Y(at)\right].$$

Evaluating this derivative at $t = 0$ gives

$$E(W) = e^0\left[bm_Y(0) + am'_Y(0)\right] = aE(Y) + b.$$

Above we used the fact that $m_Y(0) = 1$ and $m'_Y(0) = E(Y)$.

To find $V(W)$, we can find $E(W^2)$ first. Taking another derivative, we have

$$\begin{aligned}
\frac{d^2}{dt^2}m_W(t) &= \frac{d}{dt}\left\{e^{bt}\left[bm_Y(at) + am'_Y(at)\right]\right\} \\
&= be^{bt}\left[bm_Y(at) + am'_Y(at)\right] + e^{bt}\left[abm'_Y(at) + a^2m''_Y(at)\right].
\end{aligned}$$

Evaluating this derivative at $t = 0$ gives

$$\begin{aligned}
E(W^2) &= be^0\left[bm_Y(0) + am'_Y(0)\right] + e^0\left[abm'_Y(0) + a^2m''_Y(0)\right] \\
&= b[b + aE(Y)] + abE(Y) + a^2E(Y^2) \\
&= a^2E(Y^2) + 2abE(Y) + b^2.
\end{aligned}$$

From the variance computing formula, we have

$$\begin{aligned}
V(W) = E(W^2) - [E(W)]^2 &= a^2E(Y^2) + 2abE(Y) + b^2 - [aE(Y) + b]^2 \\
&= a^2E(Y^2) + 2abE(Y) + b^2 - \{a^2[E(Y)]^2 + 2abE(Y) + b^2\} \\
&= a^2E(Y^2) - a^2[E(Y)]^2 \\
&= a^2\{E(Y^2) - [E(Y)]^2\} = a^2V(Y)
\end{aligned}$$

as claimed.

3.160. We are given that $Y \sim b(n,p)$. Recall Y counts the number of successes in n Bernoulli trials. Therefore, $Y^* = n - Y$ counts the number of failures. We know $E(Y) = np$ and $V(Y) = npq$, where $q = 1 - p$.

(a) We have

$$E(Y^*) = E(n - Y) = n - E(Y) = n - np = n(1-p) = nq$$

and

$$V(Y^*) = V(n - Y) = (-1)^2 V(Y) = npq.$$

(b) The mgf of Y^* is

$$\begin{aligned} m_{Y^*}(t) = E(e^{tY^*}) = E[e^{t(n-Y)}] = e^{nt}E(e^{-tY}) = e^{nt}m_Y(-t) &= e^{nt}(q + pe^{-t})^n \\ &= (e^t)^n(q + pe^{-t})^n \\ &= (qe^t + p)^n. \end{aligned}$$

(c) The mgf in part (b) is the mgf of a binomial distribution with number of trials n and "success probability" $q = 1 - p$. Therefore, $Y^* \sim b(n, 1-p)$.
(d) $Y^* = n - Y$ counts the number of failures.
(e) I already answered this in another problem.

3.188. We are given that $Y \sim b(n,p)$. Use the definition of conditional probability; i.e.,

$$P(Y > 1|Y \geq 1) = \frac{P(Y > 1 \text{ and } Y \geq 1)}{P(Y \geq 1)} = \frac{P(Y > 1)}{P(Y \geq 1)}.$$

The last step is true because $\{Y > 1\} \subset \{Y \geq 1\}$ so that $\{Y > 1\} \cap \{Y \geq 1\} = \{Y > 1\}$. Now use the complement rule and write

$$\begin{aligned} P(Y > 1) = 1 - P(Y \leq 1) &= 1 - P(Y = 0) - P(Y = 1) \\ &= 1 - \binom{n}{0}p^0(1-p)^n - \binom{n}{1}p^1(1-p)^{n-1} \\ &= 1 - (1-p)^n - np(1-p)^{n-1}. \end{aligned}$$

Also,

$$P(Y \geq 1) - 1 - P(Y = 0) = 1 - \binom{n}{0}p^0(1-p)^n = 1 - (1-p)^n.$$

The result follows.