GROUND RULES:

- This exam contains 5 questions. Each question is worth 20 points. This exam is worth 100 points.
- This is a open-book and open-notes exam. You cannot use anything else other than the book or the notes; no internet; no discussion with anyone else except me, etc.
- Each question contains subparts. On each part, there is opportunity for partial credit, so show all of your work and explain all of your reasoning. **Translation:** No work/no explanation means no credit.
- **My preference:** (1) Print out exam if you are attending the class online. (2) Print your name legibly at the top of this page. (3) Write your solution to each problem on its own page. You can use the back of the page if needed; you can append extra pages if needed. (4) Collect your pages when done and staple in upper left corner. Keep all pages/solutions in order.
- I prefer you write out your solutions "by hand." In the past, I have noticed students who type their solutions (in LaTeX or Word) often do not provide enough detail and/or do not explain their reasoning very well.
- Your solutions should be turned in to me no later than Friday, September 18 at 5:00pm. I would prefer your solutions be delivered to me in person. If this is not possible, let me know via email.

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. Suppose P_1 and P_2 are probability set functions (on a sample space S) that satisfy the Kolmolgorov Axioms. In class, we proved

$$P(A) = \alpha P_1(A) + (1 - \alpha)P_2(A),$$

where $A \subset S$ and $0 \leq \alpha \leq 1$, satisfies the Kolmolgorov Axioms and hence is also a probability set function on S.

Do any of the following set function definitions satisfy the Kolmogorov Axioms? For each one, prove it does or show why it does not.

(a) $P(A) = P_1(A) - P_2(A)$

(b) $P(A) = \dfrac{P_1(A)}{P_1(A) + P_2(A)}$

(c) $P(A) = P_1(A)P_2(A)$

2. The State Hygienic Laboratory (SHL) at University of Iowa tests thousands of males each year for chlamydia and gonorrhea. The SHL only tests males whose specimens are shipped to them; these are usually "high-risk" subjects in the state (therefore, don't be alarmed by the probabilities which follow). In this high-risk male cohort, I have determined the probability of having chlamydia is 0.158 and the probability of having gonorrhea is 0.031. The probability of having both diseases is 0.013.

Before you get started, define two relevant events C and G (this is obvious) for a randomly selected male in this cohort and then write the probabilities above using the appropriate notation.

(a) What is the probability a high-risk male has at least one disease?
(b) What is the probability a high-risk male has chlamydia but not gonorrhea?
(c) What is the probability a high-risk male does not have gonorrhea given that he does not have chlamydia?
(d) What is the probability a high-risk male has exactly one disease given that he has at least one disease?
(e) Intuitively, these two diseases should not occur independently (their risk factors are nearly identical). Verify this using the definition of independence.

In each part above, make sure you are identifying the relevant events and using the correct probability notation in your solutions (like we do in the notes); don't just give me equations of numbers without identifying what the numbers are. Also, you can use Venn Diagrams to aid in your reasoning, but your solutions should be written more authoritatively using events and their probabilities by appealing to the rules we have derived in the class.

3. Excluding me, there are 12 professors in the Department of Statistics at University of South Carolina. Below is a list of these 12 professors stratified by rank (Full, Associate, Assistant); I have also included the gender of each faculty member.

Name	Rank	Gender
Peña	Full	M
Grego	Full	M
Huang	Full	F
Habing	Associate	M
Hitchcock	Associate	M
Wang L	Associate	M
Lin	Associate	F
Wang D	Associate	M
Ho	Assistant	F
Gregory	Assistant	M
Shin	Assistant	M
Bai	Assistant	M

Treat parts (a) and (b) below separately; do not attempt to combine them.

(a) I will send a personal email to each faculty member, and I will keep track of the order in which these 12 emails are sent. Under the assumption that each of these orderings is equally likely, what is the probability the three Full professors' emails are sent in succession (i.e., one right after the other)?

(b) In May, I was asked by my Associate Dean to form a committee of three professors to formulate criteria for faculty merit raises in my department. I was asked to include one professor at each rank.

(i) How many committees are possible? Answer this question in two ways: one where the committee positions are distinct and one where they are not.

(ii) I used R to randomly select the committee according to my Associate Dean's request (i.e., to include one professor at each rank). The committee randomly selected was Huang, Lin, and Ho−my three female colleagues. I was later asked whether the committee should have been reselected because "it would be extremely unlikely to select three females given the distribution of males and females in my department." How would you have responded to this claim?

4. Let a_n denote the number of ways of tossing a coin n times so that successive heads do not appear. One can prove that the sequence a_n obeys the recursive formula

$$a_n = a_{n-1} + a_{n-2},$$

for $n \geq 2$, where $a_0 = 1$ and $a_1 = 2$. For example, when $n = 2$, the sample space is

$$S = \{(\text{HH}), (\text{HT}), (\text{TH}), (\text{TT})\}$$

where it is clear $a_2 = a_1 + a_0 = 2 + 1 = 3$. That is, there are three outcomes in S where successive heads do not appear.

(a) Write out the sample spaces when $n = 3$ and $n = 4$ and verify the recursive formula above holds for these cases.
(b) Show that

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \frac{1+\sqrt{5}}{2}.$$

Hint: First use algebra to show

$$\frac{a_n}{a_{n-1}} = 1 + \frac{1}{\frac{a_{n-1}}{a_{n-2}}}$$

and then take the limit of both sides. Note that $\lim_{n\to\infty}(a_n/a_{n-1}) = \lim_{n\to\infty}(a_{n-1}/a_{n-2})$. Why? Call this common limit a and then solve for a.
(c) Let P_n denote the probability successive heads do not appear when a coin is tossed n times. Determine P_n (as a function of a_n) when all possible outcomes among the n tosses are equally likely. What is P_{10}?
(d) As in part (c), continue to assume all possible outcomes among the n tosses are equally likely. Using the limiting result in part (b), argue

$$P_n \approx \frac{P_{n-1}}{2}\left(\frac{1+\sqrt{5}}{2}\right)$$

when n is large.

5. A defendant tried by a 3-judge panel is declared guilty if all three judges cast votes of guilty. Consider the following information:

- 70 percent of all defendants are guilty
- given the defendant is guilty, the judges' votes are mutually independent and each judge will vote guilty with probability 0.8
- given the defendant is not guilty, the judges' votes are mutually independent and each judge will vote guilty with probability 0.2.

(a) Find the probability a randomly selected defendant is declared guilty by the panel.
(b) Suppose the defendant is guilty. What is the probability he will be declared not guilty by the panel?
(c) Find the probability that judge 3 votes guilty given that judges 1 and 2 vote guilty.

Hint: For this problem, define $G = \{\text{defendant is guilty}\}$ and $J_i = \{\text{judge } i \text{ votes guilty}\}$, for $i = 1, 2, 3$. Note that a defendant is declared guilty by the panel when $J_1 \cap J_2 \cap J_3$ occurs. Use LOTP in part (a), noting that G and $\overline{G}$ partition the sample space. In part (c), you want to calculate $P(J_3|J_1 \cap J_2)$. Do not argue this probability is $P(J_3)$ by independence; this is not correct. The three judges' votes are mutually independent only *after* you condition on the defendant's guilty status.