1. (a) This set function does not satisfy the Kolmogorov Axioms. We know $P_1(A) \geq 0$ and $P_2(A) \geq 0$. However, there is no guarantee $P_1(A) - P_2(A)$ will be nonnegative. Therefore, this set function does not even satisfy the first axiom!
(b) This set function does not satisfy the Kolmogorov Axioms. It does satisfy the first axiom, provided that either $P_1(A)$ or $P_2(A)$ are strictly larger than zero; note that $P_1(A) \geq 0$ and $P_1(A) + P_2(A) \geq 0$. Therefore,

$$P(A) = \frac{P_1(A)}{P_1(A) + P_2(A)} \geq 0$$

as well. However,

$$P(S) = \frac{P_1(S)}{P_1(S) + P_2(S)} = \frac{1}{1+1} = \frac{1}{2}.$$

Because $P(S) \neq 1$, this set function does not satisfy the second axiom.
(c) This set function does not satisfy the Kolmogorov Axioms. It does satisfy the first two axioms; note that $P_1(A) \geq 0, P_2(A) \geq 0 \Longrightarrow P(A) = P_1(A)P_2(A) \geq 0$ and $P(S) = P_1(S)P_2(S) = 1 \times 1 = 1$. However, countable additivity fails. To see why, suppose $A_1, A_2, ...,$ forms a collection of pairwise mutually exclusive events. Because P_1 and P_2 are countably additive, we know

$$P_1\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P_1(A_i) \quad \text{and} \quad P_2\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P_2(A_i).$$

However,

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) = P_1\left(\bigcup_{i=1}^{\infty} A_i\right) P_2\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{i=1}^{\infty} P_1(A_i) \sum_{i=1}^{\infty} P_2(A_i) \\ &\neq \sum_{i=1}^{\infty} P_1(A_i)P_2(A_i) \\ &= \sum_{i=1}^{\infty} P(A_i). \end{aligned}$$

The "$\neq$" above is because "the product of the two sums is not the sum of the products." Therefore, this set function does not satisfy the third axiom.

2. Define the following events:

$$\begin{aligned} C &= \{\text{high-risk male has chlamydia}\} \\ G &= \{\text{high-risk male has gonorrhea}\}. \end{aligned}$$

We are given $P(C) = 0.158$, $P(G) = 0.031$, and $P(C \cap G) = 0.013$.

(a) "At least one disease" means $C \cup G$ has occurred. By the additive rule,

$$P(C \cup G) = P(C) + P(G) - P(C \cap G) = 0.158 + 0.031 - 0.013 = 0.176.$$

(b) Having chlamydia but not gonorrhea means $C \cap \overline{G}$ has occurred. Write

$$C = (C \cap G) \cup (C \cap \overline{G}).$$

Because $C \cap G$ and $C \cap \overline{G}$ are mutually exclusive events, Axiom 3 says

$$P(C) = P(C \cap G) + P(C \cap \overline{G}) \implies P(C \cap \overline{G}) = P(C) - P(C \cap G) = 0.158 - 0.013 = 0.145.$$

(c) We want to calculate $P(\overline{G}|\overline{C})$. Using the definition of conditional probability, we have

$$P(\overline{G}|\overline{C}) = \frac{P(\overline{G} \cap \overline{C})}{P(\overline{C})}.$$

By DeMorgan's Law and the complement rule, the numerator is

$$P(\overline{G} \cap \overline{C}) = P(\overline{G \cup C}) = 1 - P(G \cup C) = 1 - 0.176 = 0.824.$$

The denominator is $P(\overline{C}) = 1 - P(C) = 1 - 0.158 = 0.842$ by the complement rule. Therefore,

$$P(\overline{G}|\overline{C}) = \frac{P(\overline{G} \cap \overline{C})}{P(\overline{C})} = \frac{0.824}{0.842} \approx 0.979.$$

(d) "Exactly one disease" means the symmetric difference $C \triangle G = (C \cap \overline{G}) \cup (\overline{C} \cap G)$ has occurred; see HW1 (Problem 2.131). We want to calculate $P(C \triangle G | C \cup G)$. From the definition of conditional probability, we have

$$P(C \triangle G | C \cup G) = \frac{P((C \triangle G) \cap (C \cup G))}{P(C \cup G)} = \frac{P(C \triangle G)}{P(C \cup G)}.$$

The last step is true because $C \triangle G \subset C \cup G$; therefore, $(C \triangle G) \cap (C \cup G) = C \triangle G$. We know $P(C \cup G) = 0.176$ from part (a). From Problem 2.131, we know

$$P(C \triangle G) = P(C) + P(G) - 2P(C \cap G) = 0.158 + 0.031 - 2(0.013) = 0.163.$$

Therefore,

$$P(C \triangle G | C \cup G) = \frac{P(C \triangle G)}{P(C \cup G)} = \frac{0.163}{0.176} \approx 0.926.$$

(e) The events C and G are independent if $P(C \cap G) = P(C)P(G)$. This is not true here; note that

$$0.013 = P(C \cap G) \neq P(C)P(G) = 0.158 \times 0.031 = 0.004898.$$

Therefore, these two diseases are not independent.

3. (a) Like the flag problem in the notes, we offer two solutions:

- Solution 1: Do not distinguish among professors within the same rank (i.e., we have 3 "groups" of professors)
- Solution 2: Treat the professors as distinct objects.

Solution 1: Conceptualize the sample space S to consist of outcomes of the form

$$(\_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_)$$

where each position is occupied by either Full (F), Associate (C), or Assistant (T). For example, the outcome

$$(\text{F C C F T T T F T C C C})$$

would mean the first email was sent to a full professor, the second email was sent to an associate professor, the third email was sent to an associate professor and so on. How many outcomes are in this S? This is the same question as "how many ways are there to permute the 12 (nondistinct) objects in {F, F, F, C, C, C, C, C, T, T, T, T}?"

$$\textit{Ans:} \quad \binom{12}{3\ 5\ 4} = \frac{12!}{3!\ 5!\ 4!} = 27720.$$

Therefore, this sample space consists of $N = 27720$ different outcomes. Now define the event

$$A = \{\text{full professor emails sent in succession}\}.$$

All we have to do is count n_a, the number of outcomes in A. We can get this answer using the basic rule of counting:

$$\begin{aligned} n_1 &= \text{number of ways to select 3 adjacent positions for full professor emails} = 10 \\ n_2 &= \text{number of ways to permute \{C, C, C, C, C, T, T, T, T\} in 9 remaining positions} \\ &= \binom{9}{5\ 4} = 126. \end{aligned}$$

Therefore,

$$n_a = n_1 \times n_2 = 10 \times 126 = 1260.$$

Assuming each outcome in S is equally likely, we have

$$P(A) = \frac{n_a}{N} = \frac{1260}{27720} \approx 0.045.$$

Solution 2: Conceptualize the sample space S to consist of outcomes of the form

$$(\_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_)$$

where each position is occupied by an integer 1, 2, ..., 12 (i.e., these are distinct objects). For example, the outcome

$$(6\ \ 8\ \ 10\ \ 12\ \ 1\ \ 3\ \ 5\ \ 7\ \ 9\ \ 11\ \ 2\ \ 4)$$

is understood to mean that Wang L gets the first email, Wang D gets the second, Shin gets the third, and so on. How many outcomes are in this S? This is the same question as "how many ways can the first twelve integers 1, 2, 3, ..., 12 be permuted?" *Ans:* 12!. Therefore, this sample space consists of $N = 12! = 479001600$ different outcomes. Now define the event

$$A = \{\text{full professor emails sent in succession}\}.$$

All we have to do is count n_a, the number of outcomes in A. We can get this answer using the basic rule of counting:

$$\begin{aligned} n_1 &= \text{number of ways to select 3 adjacent positions for full professor emails} = 10 \\ n_2 &= \text{number of ways to permute 3 full professors} = 3! \\ n_3 &= \text{number of ways to permute 9 remaining professors} = 9! \end{aligned}$$

Therefore,

$$n_a = n_1 \times n_2 \times n_3 = 10 \times 3! \times 9! = 21772800.$$

Assuming each outcome in S is equally likely, we have

$$P(A) = \frac{n_a}{N} = \frac{21772800}{479001600} \approx 0.045.$$

(b) The merit raise committee consists of 3 people. Therefore, I am conceptualizing the sample space S for this part to consist of outcomes of the form

$$(\_\_\ \_\_\ \_\_)$$

where each position is occupied by a professor. For example, one outcome might be

(Grego Wang L Ho).

(i) We need to include exactly one professor at each rank. If the committee positions are not distinct, there are $3 \times 5 \times 4 = 60$ possible committees (basic rule of counting). If the committee positions are distinct, then for each of the possible 60 committees above, there are $3! = 6$ ways to permute the 3 professors. Therefore, there are $3 \times 5 \times 4 \times 3! = 360$ possible committees if the positions are distinct.
(ii) Among the 60 possible committees (assuming positions are not distinct), there is only one committee that consists of all females, namely, the outcome

(Huang Lin Ho).

Therefore, assuming each committee is equally likely, the probability of an all-female committee is $1/60 \approx 0.017$. As for the claim, "it would be extremely unlikely to select three females....". I guess this statement is somewhat true. Certainly before the committee was selected, I would not have expected this outcome. However, even events with small probabilities will occur! In fact, there is a well known result in advanced probability theory (called the Borel-Cantelli lemma) which states that if we were to repeat this experiment over and over again, not only would this event occur eventually, but it would occur an infinite number of times!

4. (a) When $n = 3$, the sample space is

$S = \{$(HHH), (HHT), **(HTH)**, (THH), **(HTT)**, **(THT)**, **(TTH)**, **(TTT)**$\}$.

The outcomes were successive heads do not appear are shown bolded. Note that $a_3 = a_2 + a_1 = 3 + 2 = 5$. When $n = 4$, the sample space is

$S = \{$(HHHH), (HHHT), (HHTH), (HHTT), (HTHH), **(HTHT)**, **(HTTH)**, **(HTTT)**, **(TTTT)**, **(TTTH)**, **(TTHT)**, (TTHH), **(THTT)**, **(THTH)**, (THHT), (THHH)$\}$.

The outcomes were successive heads do not appear are shown bolded. Note that $a_4 = a_3 + a_2 = 5 + 3 = 8$.
(b) From the recursive formula $a_n = a_{n-1} + a_{n-2}$, note that

$$\frac{a_n}{a_{n-1}} = \frac{a_{n-1} + a_{n-2}}{a_{n-1}} = 1 + \frac{a_{n-2}}{a_{n-1}} = 1 + \frac{1}{\frac{a_{n-1}}{a_{n-2}}}.$$

Taking limits of both sides, we have

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{a_{n-1}}{a_{n-2}}}\right) = 1+\frac{1}{\lim_{n\to\infty}\frac{a_{n-1}}{a_{n-2}}}.$$

The last step is true because taking limits is preserved under continuous mappings; i.e., $f(x) = 1+1/x$ is a continuous function of x for $x>0$. We know

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\frac{a_{n-1}}{a_{n-2}} = a, \quad \text{say.}$$

These limits are equal because $n\to\infty$ and the end behavior of the sequences a_n/a_{n-1} and a_{n-1}/a_{n-2} is the same. Therefore, from the equation at the top of the page, we have

$$a = 1+\frac{1}{a} \implies a = \frac{a+1}{a} \implies a^2-a-1=0.$$

From the quadratic formula, we have

$$a = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)} = \frac{1\pm\sqrt{5}}{2}.$$

We retain the positive root

$$\frac{1+\sqrt{5}}{2}$$

because a must be positive; i.e., $a_n>0$ for all $n\geq 1$ so $a_n/a_{n-1}>0$ and hence its limit is positive. Some of you may recognize this solution for a as the *golden ratio*−interesting!

(c) For general n, there are 2^n outcomes in S. To see why, an outcome in S has the following form

$$(\underbrace{\_\_\ \_\_\ \_\_\ \cdots\cdots\ \_\_}_{n \text{ positions}}).$$

For each position, there are 2 possible outcomes (H and T). From the basic rule of counting, there are $2\times 2\times 2\times\cdots\times 2 = 2^n$ outcomes in total. If each of these 2^n outcomes is equally likely, then

$$P_n = \frac{a_n}{2^n}.$$

When $n=10$, we have $a_{10}=144$ and $2^{10}=1024$. Therefore, $P_{10} = 144/1024 \approx 0.141$.

(d) Note that

$$P_n = \frac{a_n}{2^n} \quad \text{and} \quad P_{n-1} = \frac{a_{n-1}}{2^{n-1}}.$$

Therefore,

$$\frac{P_n}{P_{n-1}} = \frac{\frac{a_n}{2^n}}{\frac{a_{n-1}}{2^{n-1}}} = \frac{a_n}{2^n}\times\frac{2^{n-1}}{a_{n-1}} = \frac{1}{2}\left(\frac{a_n}{a_{n-1}}\right).$$

However, when n is large, $a_n/a_{n-1}\approx(1+\sqrt{5})/2$ from part (b). Therefore,

$$\frac{P_n}{P_{n-1}} \approx \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right) \implies P_n \approx \frac{P_{n-1}}{2}\left(\frac{1+\sqrt{5}}{2}\right), \quad \text{for large } n.$$

5. As suggested in the hint, define $G = \{\text{defendant is guilty}\}$ and

$$\begin{aligned} J_1 &= \{\text{judge 1 votes guilty}\} \\ J_2 &= \{\text{judge 2 votes guilty}\} \\ J_3 &= \{\text{judge 3 votes guilty}\}. \end{aligned}$$

Note that G and $\overline{G} = \{\text{defendant is not guilty}\}$ partition the sample space. We have $P(G) = 0.7$ and thus $P(\overline{G}) = 0.3$ by the complement rule.

(a) A defendant is declared guilty by the panel when $J_1 \cap J_2 \cap J_3$ occurs. Therefore, we want to calculate $P(J_1 \cap J_2 \cap J_3)$. Following the hint again, let's use LOTP to write

$$P(J_1 \cap J_2 \cap J_3) = P(J_1 \cap J_2 \cap J_3|G)P(G) + P(J_1 \cap J_2 \cap J_3|\overline{G})P(\overline{G}).$$

- The second bullet says "given G, the events J_1, J_2, and J_3 are mutually independent." Therefore,

$$P(J_1 \cap J_2 \cap J_3|G) = P(J_1|G)P(J_2|G)P(J_3|G) = 0.8 \times 0.8 \times 0.8 = (0.8)^3.$$

- The third bullet says "given $\overline{G}$, the events J_1, J_2, and J_3 are mutually independent." Therefore,

$$P(J_1 \cap J_2 \cap J_3|\overline{G}) = P(J_1|\overline{G})P(J_2|\overline{G})P(J_3|\overline{G}) = 0.2 \times 0.2 \times 0.2 = (0.2)^3.$$

Finally,

$$P(J_1 \cap J_2 \cap J_3) = P(J_1 \cap J_2 \cap J_3|G)P(G) + P(J_1 \cap J_2 \cap J_3|\overline{G})P(\overline{G}) = (0.8)^3(0.7) + (0.2)^3(0.3) \approx 0.361.$$

(b) We are given the defendant is guilty; i.e., G occurs. The panel declares the defendant to be guilty when $J_1 \cap J_2 \cap J_3$ occurs. Therefore, we want to calculate $P(\overline{J_1 \cap J_2 \cap J_3}|G)$. Using the complement rule for conditional probabilities, we have

$$P(\overline{J_1 \cap J_2 \cap J_3}|G) = 1 - P(J_1 \cap J_2 \cap J_3|G) = 1 - (0.8)^3 = 0.488.$$

(c) We want to calculate $P(J_3|J_1 \cap J_2)$. Using the definition of conditional probability, we have

$$P(J_3|J_1 \cap J_2) = \frac{P(J_1 \cap J_2 \cap J_3)}{P(J_1 \cap J_2)}.$$

We already know $P(J_1 \cap J_2 \cap J_3) \approx 0.361$ from part (a). We get $P(J_1 \cap J_2)$ in the same way.

- The second bullet says "given G, the events J_1 and J_2 are independent." Therefore,

$$P(J_1 \cap J_2|G) = P(J_1|G)P(J_2|G) = 0.8 \times 0.8 = (0.8)^2.$$

- The third bullet says "given $\overline{G}$, the events J_1 and J_2 are independent." Therefore,

$$P(J_1 \cap J_2|\overline{G}) = P(J_1|\overline{G})P(J_2|\overline{G}) = 0.2 \times 0.2 = (0.2)^2.$$

From the LOTP, we have

$$P(J_1 \cap J_2) = P(J_1 \cap J_2|G)P(G) + P(J_1 \cap J_2|\overline{G})P(\overline{G}) = (0.8)^2(0.7) + (0.2)^2(0.3) = 0.46.$$

Finally,

$$P(J_3|J_1 \cap J_2) = \frac{P(J_1 \cap J_2 \cap J_3)}{P(J_1 \cap J_2)} \approx \frac{0.361}{0.46} \approx 0.785.$$