GROUND RULES:

- This exam contains 5 questions. Each question is worth 20 points. This exam is worth 100 points.
- This is a open-book and open-notes exam. You cannot use anything else other than the book or the notes; no internet; no discussion with anyone else except me, etc.
- Each question contains subparts. On each part, there is opportunity for partial credit, so show all of your work and explain all of your reasoning. **Translation:** No work/no explanation means no credit.
- If you use R to answer any part or to check your work, please include all code and output as attachments. Do not just write out the code you used.
- **My preference:** (1) Print out exam if you are attending the class online. (2) Print your name legibly at the top of this page. (3) Write your solution to each problem on its own page. You can use the back of the page if needed; you can append extra pages if needed. (4) Collect your pages when done and staple in upper left corner. Keep all pages/solutions in order.
- I prefer you write out your solutions "by hand." In the past, I have noticed students who type their solutions (in LaTeX or Word) often do not provide enough detail and/or do not explain their reasoning very well.
- Your solutions should be turned in to me no later than Thursday, November 5 at 4:00pm. **Note this change in time.** I would prefer your solutions be delivered to me in person. If this is not possible, let me know via email.

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. Suppose Y is a continuous random variable with support on the whole real line; i.e., $R = \mathbb{R} = (-\infty, \infty)$. Let $f_Y(y)$ denote the probability density function (pdf) of Y, and let $F_Y(y)$ denote the cumulative distribution function (cdf) of Y.

(a) Prove that

$$F_Y(y+1) - F_Y(y)$$

is a valid pdf. *Hint:* It suffices to prove $F_Y(y+1) - F_Y(y) \geq 0$ and that $F_Y(y+1) - F_Y(y)$ integrates to 1 over $\mathbb{R}$.

Suppose $a \in \mathbb{R}$ and $b > 0$; i.e., these values are both constants, and define

$$X = \frac{Y - a}{b}.$$

(b) Explain why X is also a continuous random variable. Your explanation does not have to be mathematically rigorous, but it should be convincing (to me).
(c) Let $F_X(x)$ denote the cdf of X. Prove that

$$F_X(x) = F_Y(a + bx)$$

and thus show the pdf of X, where nonzero, is

$$f_X(x) = bf_Y(a + bx).$$

2. The continuous random variable Y has probability density function (pdf)

$$f_Y(y) = \begin{cases} \frac{1}{2}(1+y), & -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the cumulative distribution function (cdf) of Y. Remember the cdf is a function defined for all values of $y \in (-\infty, \infty)$, not just for those values of y in the support.
(b) Show the moment generating function (mgf) of Y is

$$m_Y(t) = \frac{(2t-1)e^t + e^{-t}}{2t^2},$$

for $t \neq 0$. How should the $m_Y(t)$ be defined to make it continuous at $t = 0$?
(c) Calculate $E(Y)$ and $V(Y)$.

3. An engineer is studying the reliability of dishwasher motors. For a certain brand of motor, the engineer models the time until failure Y (in years) as an exponential random variable with mean $\beta = 10$.

(a) What is the probability a dishwater motor of this brand does not fail in its first 15 years of operation?
(b) For this brand of motor, the 80th percentile of Y is about 16.1 years. Provide a mathematical argument showing this is correct (don't give me R code/output as an argument on this part unless it is to check your work). In addition, draw a picture showing where this number comes from.
(c) The cost of this brand of motor is 200 dollars. The company which manufacturers the dishwasher guarantees the following warranty to its customers:

- If the motor fails in the first year of operation (i.e., $Y \leq 1$), the customer is reimbursed 200 dollars (full price of replacement).
- If the motor fails in the second or third year of operation (i.e., $1 < Y \leq 3$), the customer is reimbursed 100 dollars (half price of replacement).
- If the motor fails after the third year of operation (i.e., $Y > 3$), then the reimbursement is 0 dollars.

Let the random variable R denote the amount of the reimbursement for a randomly selected customer. Find the variance of R.

4. A random variable Y has gamma distribution with shape parameter $\alpha = 3$ and scale parameter $\beta > 0$; i.e., the pdf of Y is

$$f_Y(y) = \begin{cases} cy^2 e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.
(b) Derive a general formula for the mth moment of Y; i.e., for $E(Y^m)$. Simplify your answer as much as possible. Do **not** use the moment generating function of Y.
(c) At my voting location today, suppose President Trump supporters arrive to vote according to a Poisson process with mean 10 per hour. In any given hour, what is the probability it will take longer than 15 minutes for the third President Trump supporter to arrive? Note that 15 minutes = 0.25 hour.

5. Each day, the United States Postal Service records Y, the proportion of "Priority Mail Express" packages that are delivered late. The probability density function for Y is proportional to $y(1-y)^5$, that is,

$$f_Y(y) = \begin{cases} cy(1-y)^5, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.
(b) Find the mean, median, and mode of Y. *Note:* If you cannot calculate the median exactly, set up an equation that when solved will give the median (for partial credit). The mode of Y is the value of y which maximizes $f_Y(y)$.
(c) The *odds* that a "Priority Mail Express" package is delivered late is a function of Y, specifically,

$$g(Y) = \frac{Y}{1-Y}.$$

Find the expected odds; i.e., the expected value of $g(Y)$.