1. (a) We know $F_Y(y+1) \geq F_Y(y)$ for all $y \in \mathbb{R}$ because $F_Y(y)$ is nondecreasing (one of the properties of a cdf). Therefore, $F_Y(y+1) - F_Y(y) \geq 0$. To show the second condition, recall that

$$F_Y(y) = \int_{-\infty}^{y} f_Y(x)dx.$$

Therefore,

$$F_Y(y+1) = \int_{-\infty}^{y+1} f_Y(x)dx$$

and hence

$$F_Y(y+1) - F_Y(y) = \int_{-\infty}^{y+1} f_Y(x)dx - \int_{-\infty}^{y} f_Y(x)dx = \int_{y}^{y+1} f_Y(x)dx.$$

We now show $F_Y(y+1) - F_Y(y)$ integrates to 1; i.e.,

$$\int_{-\infty}^{\infty} [F_Y(y+1) - F_Y(y)]dy = \int_{y=-\infty}^{y=\infty} \left(\int_{x=y}^{x=y+1} f_Y(x)dx \right) dy = 1.$$

In the last (double) integral, we are integrating over this region in $\mathbb{R}^2$:

$$y \leq x \leq y+1 \qquad -\infty < y < \infty.$$

This is the same region as

$$-\infty < x < \infty \qquad x - 1 \leq y \leq x.$$

Therefore, interchanging the order of integration (Fubini's Theorem), the last (double) integral equals

$$\begin{aligned}
\int_{x=-\infty}^{x=\infty} \left(\int_{y=x-1}^{y=x} f_Y(x)dy \right) dx &= \int_{x=-\infty}^{x=\infty} f_Y(x) \left(\int_{y=x-1}^{y=x} 1dy \right) dx \\
&= \int_{-\infty}^{\infty} f_Y(x)[x - (x-1)]dx = \int_{-\infty}^{\infty} f_Y(x)dx = 1.
\end{aligned}$$

The last step is true because the pdf of Y integrates to 1.

(b) The support of Y is $\mathbb{R}$, that is, the possible values of Y are $y \in (-\infty, \infty)$. Now consider

$$x = \frac{y-a}{b} = \left(\frac{1}{b}\right) y - \frac{a}{b},$$

a linear function of y. If $y \in (-\infty, \infty)$, then clearly $x \in (-\infty, \infty)$ too. Therefore, the possible values of X is an uncountable set; i.e., the random variable X is continuous.

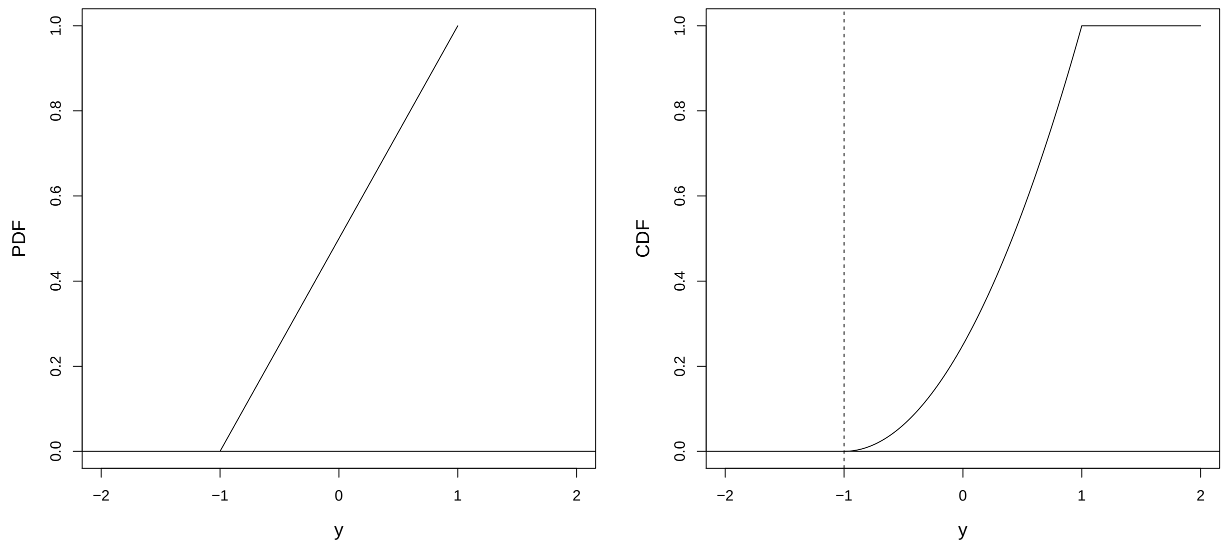
(c) The cdf of X is

$$F_X(x) = P(X \leq x) = P\left(\frac{Y-a}{b} \leq x\right) = P(Y \leq a + bx) = F_Y(a+bx).$$

Taking derivatives, the pdf of X is

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}F_Y(a+bx) = f_Y(a+bx) \times \underbrace{\frac{d}{dx}(a+bx)}_{\text{chain rule}} = bf_Y(a+bx),$$

as claimed.



2. (a) The pdf of Y is shown above (left). To derive the cdf $F_Y(y)$, we consider three cases:

Case 1: When $y \leq -1$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{y} 0dt = 0.$$

Case 2: When $-1 < y < 1$,

$$\begin{aligned} F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt &= \int_{-\infty}^{-1} 0dt + \int_{-1}^{y} \frac{1}{2}(1+t)dt \\ &= 0 + \frac{1}{2}\left(t + \frac{t^2}{2}\right)\Bigg|_{t=-1}^{y} = \frac{1}{2}\left[\left(y + \frac{y^2}{2}\right) - \left(-1 + \frac{(-1)^2}{2}\right)\right] = \frac{y^2}{4} + \frac{y}{2} + \frac{1}{4}. \end{aligned}$$

Case 3: When $y \geq 1$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{-1} 0dt + \underbrace{\int_{-1}^{1} \frac{1}{2}(1+t)dt}_{=1} + \int_{1}^{y} 0dt = 1.$$

Summarizing, the cdf of Y is

$$F_Y(y) = \begin{cases} 0, & y \leq -1 \\ \frac{y^2}{4} + \frac{y}{2} + \frac{1}{4}, & -1 < y < 1 \\ 1, & y \geq 1. \end{cases}$$

The cdf of Y is shown above (right).
(b) The mgf of Y is

$$\begin{aligned} m_Y(t) = E(e^{tY}) = \int_{\mathbb{R}} e^{ty} f_Y(y)dy = \int_{-1}^{1} e^{ty} \times \frac{1}{2}(1+y)dy &= \frac{1}{2}\int_{-1}^{1} (e^{ty} + ye^{ty})dy \\ &= \frac{1}{2}\left[\int_{-1}^{1} e^{ty}dy + \int_{-1}^{1} ye^{ty}dy\right]. \end{aligned}$$

The first integral above is

$$\int_{-1}^{1} e^{ty} dy = \left.\frac{e^{ty}}{t}\right|_{-1}^{1} = \frac{e^t}{t} - \frac{e^{-t}}{t}.$$

The second integral above

$$\int_{-1}^{1} ye^{ty} dy$$

can be done by using integration by parts. Let

$$\begin{aligned} u &= y & du &= dy \\ dv &= e^{ty}dy & v &= \frac{e^{ty}}{t}. \end{aligned}$$

With these selections, the second integral equals

$$\left.\frac{ye^{ty}}{t}\right|_{-1}^{1} - \int_{-1}^{1} \frac{e^{ty}}{t} dy = \left(\frac{e^t}{t} + \frac{e^{-t}}{t}\right) - \frac{1}{t}\int_{-1}^{1} e^{ty} dy = \left(\frac{e^t}{t} + \frac{e^{-t}}{t}\right) - \frac{1}{t}\left(\frac{e^t}{t} - \frac{e^{-t}}{t}\right).$$

Therefore,

$$\begin{aligned} m_Y(t) = \frac{1}{2}\left[\frac{e^t}{t} - \frac{e^{-t}}{t} + \left(\frac{e^t}{t} + \frac{e^{-t}}{t}\right) - \frac{1}{t}\left(\frac{e^t}{t} - \frac{e^{-t}}{t}\right)\right] &= \frac{1}{2}\left(\frac{2e^t}{t} - \frac{e^t}{t^2} + \frac{e^{-t}}{t^2}\right) \\ &= \frac{1}{2t^2}\left(2te^t - e^t + e^{-t}\right) \\ &= \frac{(2t-1)e^t + e^{-t}}{2t^2}, \end{aligned}$$

as claimed. This is the mgf of Y provided that $t \neq 0$. When $t = 0$, we need to define $m_Y(t)$ in a way so that it is continuous at that point. Therefore, let's calculate

$$\lim_{t\to 0} m_Y(t) = \lim_{t\to 0} \frac{(2t-1)e^t + e^{-t}}{2t^2}.$$

Plugging in $t = 0$ produces "0/0," an indeterminate form. Therefore, let's use L'Hôpital's Rule:

$$\lim_{t\to 0} \frac{(2t-1)e^t + e^{-t}}{2t^2} = \lim_{t\to 0} \frac{2e^t + (2t-1)e^t - e^{-t}}{4t}.$$

Plugging in $t = 0$ produces "0/0" again, so let's use L'Hôpital's Rule again:

$$\lim_{t\to 0} \frac{2e^t + (2t-1)e^t - e^{-t}}{4t} = \lim_{t\to 0} \frac{2e^t + 2e^t + (2t-1)e^t + e^{-t}}{4} = \frac{4}{4} = 1.$$

Therefore,

$$m_Y(t) = \begin{cases} \dfrac{(2t-1)e^t + e^{-t}}{2t^2}, & t \neq 0 \\ 1, & t = 0. \end{cases}$$

(c) The easy way to get $E(Y)$ and $V(Y)$ is to just work with the pdf directly. We have

$$\begin{aligned} E(Y) = \int_{\mathbb{R}} y f_Y(y) dy = \int_{-1}^{1} y \times \frac{1}{2}(1+y) dy &= \frac{1}{2}\int_{-1}^{1} (y + y^2) dy \\ &= \left.\frac{1}{2}\left(\frac{y^2}{2} + \frac{y^3}{3}\right)\right|_{-1}^{1} \\ &= \frac{1}{2}\left[\left(\frac{1}{2} + \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)\right] = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}. \end{aligned}$$

The second moment of Y is

$$E(Y^2) = \int_{\mathbb{R}} y^2 f_Y(y)dy = \int_{-1}^{1} y^2 \times \frac{1}{2}(1+y)dy = \frac{1}{2}\int_{-1}^{1}(y^2+y^3)dy$$
$$= \frac{1}{2}\left(\frac{y^3}{3}+\frac{y^4}{4}\right)\Bigg|_{-1}^{1}$$
$$= \frac{1}{2}\left[\left(\frac{1}{3}+\frac{1}{4}\right)-\left(-\frac{1}{3}+\frac{1}{4}\right)\right] = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}.$$

From the variance computing formula, we have

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{2}{9}.$$

3. (a) Recall the cdf of $Y \sim$ exponential($\beta = 10$) is given by

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y/10}, & y > 0. \end{cases}$$

We want to calculate $P(Y > 15)$, the probability a motor does not fail in the first 15 years of operation. From the complement rule, we have

$$P(Y > 15) = 1 - P(Y \leq 15) = 1 - F_Y(15) = 1 - \left(1 - e^{-15/10}\right) = e^{-1.5} \approx 0.223.$$

To check our work in R,

```
> 1-pexp(15,1/10)
[1] 0.2231302
```

You could also get this answer by working with the pdf and calculating

$$P(Y > 15) = \int_{15}^{\infty} \frac{1}{10}e^{-y/10}dy \quad \text{or} \quad 1 - \int_{0}^{15} \frac{1}{10}e^{-y/10}dy.$$

The nice thing about the exponential distribution is that the cdf exists in closed form, so you don't have to.

(b) The 80th percentile (or 0.8 quantile) $\phi_{0.8}$ is found by solving

$$F_Y(\phi_{0.8}) = 1 - e^{-\phi_{0.8}/10} \overset{\text{set}}{=} 0.8 \implies e^{-\phi_{0.8}/10} = 0.2 \implies -\frac{\phi_{0.8}}{10} = \ln(0.2)$$

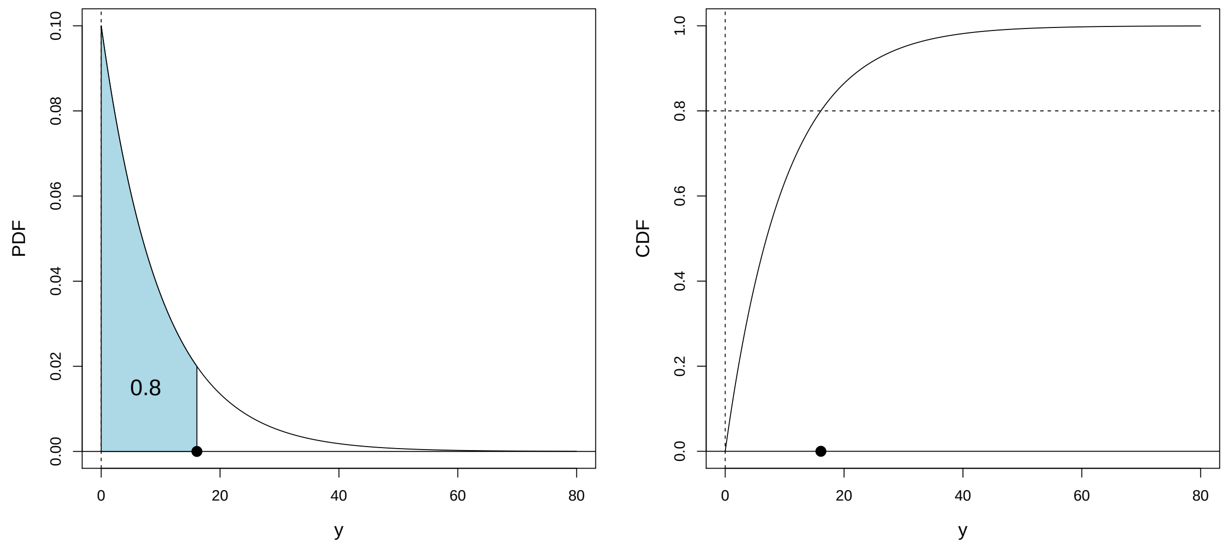
Solving for $\phi_{0.8}$ gives

$$\phi_{0.8} = -10\ln(0.2) \approx 16.1 \text{ years}.$$

You could also get this answer by working with the pdf and solving either equation below:

$$\int_{0}^{\phi_{0.8}} \frac{1}{10}e^{-y/10}dy \overset{\text{set}}{=} 0.8 \quad \text{or} \quad \int_{\phi_{0.8}}^{\infty} \frac{1}{10}e^{-y/10}dy \overset{\text{set}}{=} 0.2.$$

Either one will give you the same answer. Checking our work in R,




```
> qexp(0.8,1/10)
[1] 16.09438
```

Both pictures at the top of the page show where $\phi_{0.8}$ comes from. The pdf of Y is shown on the left and $\phi_{0.8} \approx 16.1$ is shown using a dark circle. The cdf of Y is shown on the right; $\phi_{0.8} \approx 16.1$ is shown using a dark circle.

(c) The first thing to note is that R is a discrete random variable. It can assume values 0, 100, and 200. Therefore, all we have to do is work out the probabilities of each value and then we can write the pmf of R. We have

$$\begin{aligned} P(R = 200) &= P(Y \leq 1) = F_Y(1) = 1 - e^{-1/10} \approx 0.095 \\ P(R = 100) &= P(1 < Y \leq 3) = F_Y(3) - F_Y(1) = (1 - e^{-3/10}) - (1 - e^{-1/10}) \approx 0.164. \end{aligned}$$

Therefore,

$$P(R = 0) = 1 - P(R = 200) - P(R = 100) \approx 1 - 0.095 - 0.164 = 0.741.$$

The pmf of R is shown in the following table below (probabilities to 3 dp):

r	0	100	200
$p_R(r)$	0.741	0.164	0.095

We have

$$\begin{aligned} E(R) &= 0(0.741) + 100(0.164) + 200(0.095) = 35.4 \\ E(R^2) &= 0^2(0.741) + 100^2(0.164) + 200^2(0.095) = 5440. \end{aligned}$$

From the variance computing formula, we have

$$V(R) = E(R^2) - [E(R)]^2 = 5440 - (35.4)^2 = 4186.84 \text{ (dollars)}^2$$

4. (a) If $Y \sim \text{gamma}(3, \beta)$, then

$$c = \frac{1}{\Gamma(3)\beta^3} = \frac{1}{2\beta^3}.$$

You could also get this answer by setting

$$\int_0^\infty cy^2 e^{-y/\beta} dy = 1$$

and solving for c, but this would be a lot of work (i.e., integration by parts twice).

(b) The mth moment of Y is

$$\begin{aligned} E(Y^m) = \int_{\mathbb{R}} y^m f_Y(y) dy = \int_0^\infty y^m \times \frac{1}{2\beta^3} y^2 e^{-y/\beta} dy &= \frac{1}{2\beta^3} \int_0^\infty y^{m+2} e^{-y/\beta} dy \\ &= \frac{1}{2\beta^3} \int_0^\infty y^{(m+3)-1} e^{-y/\beta} dy. \end{aligned}$$

We recognize $y^{(m+3)-1}e^{-y/\beta}$ as a gamma kernel with shape parameter $\alpha = m + 3$ and scale parameter β. Because we are integrating this function over $(0, \infty)$, we have

$$\int_0^\infty y^{(m+3)-1} e^{-y/\beta} dy = \Gamma(m+3)\beta^{m+3}.$$

Therefore,

$$E(Y^m) = \frac{1}{2\beta^3} \times \Gamma(m+3)\beta^{m+3} = \frac{\Gamma(m+3)}{2}\beta^m = \frac{(m+2)!}{2}\beta^m.$$

(c) We know the time until the third President Trump voter arrives follows a gamma distribution with shape parameter $\alpha = 3$ and scale parameter $\beta = 1/10$, the reciprocal of the Poisson mean. Therefore, if T denotes the time until the third President Trump voter arrives, then $T \sim \text{gamma}(3, 1/10)$. The nonzero part of the pdf of T is

$$\frac{1}{2\left(\frac{1}{10}\right)^3} t^2 e^{-t/\left(\frac{1}{10}\right)} = 500t^2 e^{-10t}$$

We want to calculate

$$P(T > 0.25) = 1 - P(T \leq 0.25) = 1 - \int_0^{0.25} 500t^2 e^{-10t} dt$$

I'm too tired to do this integral by hand (integration by parts twice). In R, this integral could be calculated using the `pgamma` function; i.e.,

```
> pgamma(0.25,3,10)
[1] 0.4561869
```

You could also do this integral numerically in R by using the `integrate` function; i.e.,

```
> integrand <- function(t){500*t^2*exp(-10*t)}
> integrate(integrand,lower=0,upper=0.25)
0.4561869 with absolute error < 5.1e-15
```

Therefore,

$$P(T > 0.25) = 1 - P(T \leq 0.25) \approx 1 - 0.456 = 0.544.$$

5. (a) We recognize $y(1-y)^5$ as the kernel of a beta(2, 6) pdf; i.e.,

$$y(1-y)^5 = y^{2-1}(1-y)^{6-1}.$$

Because the pdf of Y has support over $(0,1)$, we know $Y \sim \text{beta}(2,6)$ and hence

$$c = \frac{\Gamma(8)}{\Gamma(2)\Gamma(6)} = \frac{7!}{1!\ 5!} = 42.$$

You could also get this answer by setting

$$\int_0^1 cy(1-y)^5 dy = 1$$

and solving for c.
(b) The mean of $Y \sim \text{beta}(2,6)$ is

$$E(Y) = \frac{2}{2+6} = 0.25.$$

Therefore, the expected proportion of "Priority Mail Express" packages arriving late is 0.25. Note that you could also get this answer by calculating

$$\int_0^1 y \times 42y(1-y)^5 dy.$$

The median of Y, $\phi_{0.5}$, could be obtained by solving

$$\int_0^{\phi_{0.5}} 42y(1-y)^5 dy \stackrel{\text{set}}{=} 0.5 \quad \text{or} \quad \int_{\phi_{0.5}}^1 42y(1-y)^5 dy \stackrel{\text{set}}{=} 0.5$$

for $\phi_{0.5}$. This would be difficult to do "by hand," and here is why. The integrand is a polynomial of degree 6; therefore, the antiderivative of $42y(1-y)^5$ would be a polynomial of degree 7. After using the FTOC, this would leave us to find the 7 roots of a polynomial (in $\phi_{0.5}$) of degree 7 and then isolating the one between 0 and 1. The `qbeta` function in R is the only way I know how to get the answer easily; i.e.,

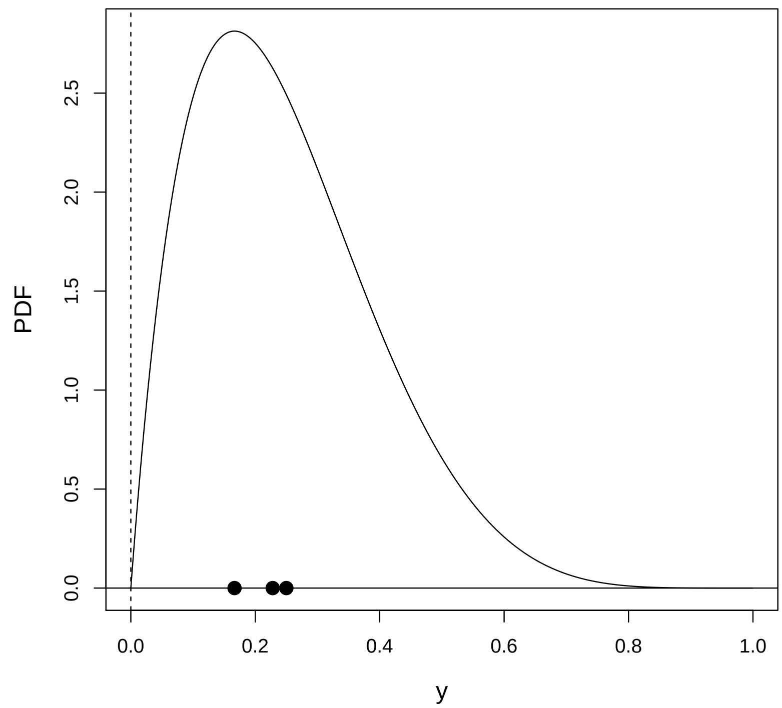
```
> qbeta(0.5,2,6)
[1] 0.22849
```

For a continuous random variable, the mode of Y is defined to be the value of y that maximizes $f_Y(y)$. This is a straightforward calculus problem. Using the product rule, the derivative of $f_Y(y)$ is

$$\begin{aligned} \frac{d}{dy} f_Y(y) = \frac{d}{dy}\left[42y(1-y)^5\right] = 42\left[(1-y)^5 + 5y(1-y)^4(-1)\right] &= 42(1-y)^4[(1-y) - 5y] \\ &= 42(1-6y)(1-y)^4. \end{aligned}$$

For $0 < y < 1$, this derivative can only be zero when

$$1 - 6y = 0 \implies y = \frac{1}{6}.$$



Therefore, $y = 1/6$ is a first-order critical point of $f_Y(y)$. To verify that this maximizes $f_Y(y)$, we can use the second derivative test. We have

$$\begin{aligned}\frac{d^2}{dy^2}f_Y(y) = \frac{d^2}{dy^2}\left[42y(1-y)^5\right] &= 42\left[(-6)(1-y)^4 + 4(1-6y)(1-y)^3(-1)\right]\\ &= 42(1-y)^3(-6+6y-4+24y)\\ &= 42(1-y)^3(30y-10)\\ &= 420(1-y)^3(3y-1).\end{aligned}$$

Evaluating $(d^2/dy^2)f_Y(y)$ at the first-order critical point $y = 1/6$ gives

$$420\left(1-\frac{1}{6}\right)^3\left[3\left(\frac{1}{6}\right)-1\right] < 0.$$

Therefore, $f_Y(y)$ is concave down at $y = 1/6$; i.e., $y = 1/6$ maximizes $f_Y(y)$. The pdf of Y is shown above; the mean (0.25), median (0.228), and mode (1/6) are shown using dark circles.
(c) The expected odds is

$$\begin{aligned}E[g(Y)] = \int_{\mathbb{R}} g(y)f_Y(y)dy &= \int_0^1 \left(\frac{y}{1-y}\right)42y(1-y)^5dy\\ &= 42\int_0^1 y^2(1-y)^4dy\\ &= 42\int_0^1 y^{3-1}(1-y)^{5-1}dy = 42\times\frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} = \frac{42(2)(24)}{7!} = 0.4.\end{aligned}$$

Note that $y^2(1-y)^4 = y^{3-1}(1-y)^{5-1}$ is the kernel of a beta$(3,5)$ pdf, so the integral above is easy to do.