GROUND RULES:

- Print your name at the top of this page.
- This is an open-book and open-notes exam. You cannot use R, the Internet, or anything else other than our course materials and our text book. You cannot talk to anyone else about the exam besides me.
- You may use a calculator. **Translation:** Show all of your work; use a calculator only to do final calculations and/or to check your work.
- This exam contains 15 questions. Each question is worth 10 points. This exam is worth 150 points.
- On each question, there is opportunity for partial credit, so show all of your work and explain all of your reasoning. **Translation:** No work/no explanation means no credit.
- You have 3 hours to complete this exam.

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. Suppose A, B, and C are events in a sample space with $0 < P(A) < 1$, $0 < P(B) < 1$, and $0 < P(C) < 1$.

Prove: If B and C are independent, then

$$P(A|B) = P(A|B \cap C)P(C) + P(A|B \cap \overline{C})P(\overline{C}).$$

2. An engineer models Y, the number of cracks in defective jet engine fan blades, using the following probability mass function (pmf):

$$p_Y(y) = \begin{cases} \dfrac{y}{(y+1)!}, & y = 1, 2, 3, ... \\ 0, & \text{otherwise.} \end{cases}$$

Prepare a graph of the cumulative distribution function (cdf) of Y. Neatness counts.

3. Suppose Y has a Poisson distribution with mean $E(Y) = \lambda$; i.e., the probability mass function (pmf) of Y is

$$p_Y(y) = \begin{cases} \dfrac{\lambda^y e^{-\lambda}}{y!}, & y = 0, 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

(a) For $k \geq 1$, show that

$$E(Y^k) = \lambda E[(Y+1)^{k-1}].$$

(b) Using the result in part (a) or otherwise, show that

$$E(Y^3) = \lambda^3 + 3\lambda^2 + \lambda.$$

Hint: Do not try to use moment generating functions in part (a).

4. A seismologist models the magnitude of "minor" earthquakes as a continuous random variable Y with the following probability density function (pdf):

$$f_Y(y) = \begin{cases} \frac{1}{20}(3+y), & 0 < y < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\phi_{0.25}$, the 0.25 quantile of Y. Another name for the 0.25 quantile is the 25th percentile.

5. Let Y_1 and Y_2 denote the values of two stocks at the end of a one-year period. Suppose the joint probability density function (pdf) of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \dfrac{1}{10y_1}, & 0 < y_2 < y_1 < 10 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate both marginal pdfs. Make sure to state the support for each.

6. An electrical device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first one has failed. The device fails when the second circuit fails.

Let Y_1 and Y_2 denote the times at which the first and second circuits fail, respectively. The joint probability density function (pdf) of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 15e^{-2y_1}e^{-3y_2}, & 0 < y_1 < y_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $E(Y_2)$, the expected time at which the device will fail.

7. Gini's index is a number economists use to measure the degree of inequality in an income distribution. Let "income" be represented by the continuous random variable Y and suppose the cumulative distribution function (cdf) of Y is $F_Y(y)$.

Gini's index is given by

$$g = \frac{1}{\mu} \int_0^{\infty} F_Y(y)[1 - F_Y(y)]dy,$$

where $\mu = E(Y)$. Calculate g when Y has a uniform distribution from a to b, where $0 < a < b < \infty$.

8. A certain town has 8 hotels. If 5 people check into hotels in a day, what is the probability each person checks into a different hotel? What assumptions are you making?

9. Suppose Y has a gamma distribution with parameters α and β; i.e., the probability density function (pdf) of Y is

$$f_Y(y) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

When $\alpha > 1$, prove the mode of Y is

$$(\alpha - 1)\beta.$$

Hint: Recall the mode of a continuous random variable Y is the value of y where the pdf $f_Y(y)$ is maximized.

10. Allen, Ben, and Chris go target shooting. Suppose Allen hits the target with probability p_1, Ben hits the target with probability p_2, and Chris hits the target with probability p_3. The three probabilities are not necessarily equal.

All three shooters shoot once at the target. What is the probability exactly two shooters hit the target given that at least one shooter hits the target? What assumptions are you making?

11. The Department of Statistics at University of South Carolina has 32 graduate students: 4 MS students and 28 PhD students. I will select a sample of 2 graduate students (at random and without replacement) and record

$$Y = \text{ number of MS students (out of 2).}$$

Find the moment generating function (mgf) of Y.

12. The South Carolina Department of Revenue estimates that 5 percent of all individual state income tax forms filed in 2020 contain “serious errors.”

An auditor plans to review tax forms until he finds the first one with “serious errors.” What is the probability he will review at least 25 forms? What assumptions are you making?

13. In an early-phase clinical trial, physicians are unsure about what proportion of patients Y will respond to a new intervention. To incorporate this uncertainty, they model Y as a continuous random variable with probability density function (pdf)

$$f_Y(y) = \begin{cases} -\ln y, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the second moment of Y.

14. The random variables X and Y have the following characteristics:

$$\begin{array}{ll} E(X) = 1 & E(Y) = 5 \\ V(X) = 4 & V(Y) = 9 \end{array}$$

and $E(XY) = 3$. Find the correlation of $U_1 = X + 3Y$ and $U_2 = 2X - Y$.

15. Suppose Y_1 and Y_2 are continuous random variables with joint probability density function (pdf)

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{1}{5}(y_1+2y_2), & 0<y_1<1,\ 0<y_2<2 \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability Y_2 will be larger than Y_1.