1. Suppose $Y \sim \mathcal{U}(0,2)$ so that the probability density function (pdf) of Y is

$$f_Y(y) = \begin{cases} \frac{1}{2}, & 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the pdf of $U = Y^4 + 1$. Make sure to note the support.
(b) Find $E(U)$ and $V(U)$.
(c) Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from $f_Y(y)$. Derive the pdf of the maximum order statistic $Y_{(n)}$ and find $E(Y_{(n)})$.

2. Actuarial data are used to formulate a probability model for Y, the insurance claim amount (in \$1000s) for a certain class of customers. Suppose Y follows an exponential distribution with mean $\beta = 5$. The next $n = 10$ claims will be observed. Assume the claims are mutually independent.
(a) What is the probability the minimum of the 10 claims exceeds \$2,000?
(b) Derive the pdf of the third smallest claim amount among the 10 claims.

3. Suppose Y_1 has a gamma distribution with parameters $\alpha = 2$ and $\beta = 1$. Suppose Y_2 has an exponential distribution with mean 1. Suppose Y_1 and Y_2 are independent.
(a) Derive the probability density function of

$$U = \frac{Y_1}{Y_2}.$$

Hint: Use either the method of distribution functions or a bivariate transformation.
(b) Find $E(U)$.

4. A city has two hospitals. For a certain group of patients, the length of stay in Hospital 1 is normally distributed with mean 4.6 days and standard deviation 0.9 days. The length of stay in Hospital 2 is normally distributed with mean 4.9 days and standard deviation 1.2 days. Assume the length of stay in Hospital 1 is independent of the length of stay in Hospital 2.
(a) Find the probability a patient's stay at Hospital 1 would be longer than his/her stay at Hospital 2.
(b) Suppose independent random samples of $n_1 = n_2 = 4$ patients are taken from each hospital (that is, within each hospital, the 4 patient lengths of stay are iid, and the two samples from different hospitals are independent). Find the probability the average stay at Hospital 1 would be longer than the average stay at Hospital 2.

5. You should know the following result:

$$Y \sim \mathcal{N}(0,1) \implies U = Y^2 \sim \chi^2(1).$$

(a) Prove this result by using the method of moment generating functions. That is, derive $m_U(t)$ and show that it matches the mgf of a χ^2 random variable with 1 degree of freedom.
(b) Suppose $Y_1, Y_2, ..., Y_n$ is an iid $\mathcal{N}(0,1)$ sample. Derive the distribution of $T = \sum_{i=1}^{n} Y_i^2$. What is the distribution of $\frac{1}{n}\sum_{i=1}^{n} Y_i^2$?

6. Suppose $Y \sim \text{gamma}(\alpha, \beta)$. Define $U = h(Y) = Y^2 - 1$.
(a) Find the probability density function (pdf) of U. Make sure to note the support.
(b) Find $E(U)$ and $V(U)$.

7. Prove the following three results.

(a) If $Y_1, Y_2, ..., Y_n$ are iid Poisson(θ), then

$$U = Y_1 + Y_2 + \cdots + Y_n \sim \text{Poisson}(n\theta).$$

(b) If $Y_1, Y_2, ..., Y_n$ are independent $\mathcal{N}(\mu_i, \sigma_i^2)$, then

$$U = \sum_{i=1}^{n} \left(\frac{Y_i - \mu_i}{\sigma_i} \right)^2 \sim \chi^2(n).$$

(c) If $Y_1, Y_2, ..., Y_n$ are iid exponential(β) random variables, then

$$U = a(Y_1 + Y_2 + \cdots + Y_n) \sim \text{gamma}(n, a\beta), \quad \text{for } a > 0.$$

8. An insurance claim amount (Y, measured in \$10,000s) follows a Weibull distribution with cumulative distribution function (cdf)

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y^2}, & y > 0. \end{cases}$$

Suppose $n = 5$ claims $Y_1, Y_2, ..., Y_5$ are made. Treat $Y_1, Y_2, ..., Y_5$ as an iid sample from $F_Y(y)$.
(a) Find the probability density function (pdf) of the minimum order statistic $Y_{(1)}$.
(b) What is the probability the minimum of the claims exceeds \$5,000?

9. According to Newton's Law of Gravitation, if two bodies are a distance R apart, then the gravitational force U exerted by one body on the other is given by

$$U = h(R) = \frac{k}{R^2},$$

where k is a positive constant. Suppose R is a random variable with probability density function

$$f_R(r) = \begin{cases} 6r^5, & 0 < r < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the pdf of U. Make sure you note the support.
(b) Find $E(U)$ and $E(U^{-1})$.

10. Suppose Y_1, Y_2 is an iid sample of size $n = 2$ from a gamma$(2, 1)$ distribution. Recall the gamma$(2, 1)$ probability density function (pdf) is given by

$$f_Y(y) = \begin{cases} ye^{-y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Use the method of moment generating functions to show $U = Y_1 + Y_2 \sim \text{gamma}(4, 1)$.
(b) Prove the result in part (a) by using either the cumulative distribution function (cdf) technique or the bivariate transformation technique.

11. Suppose Y_1 and Y_2 are random variables with joint probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 4y_1y_2, & 0 < y_1 < 1,\ 0 < y_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the pdf of $U = Y_1^2 Y_2$.
(b) Find $E(U)$ and $V(U)$.

12. Suppose Y has an exponential distribution with mean $\beta = 1$; that is, the probability density function (pdf) of Y is given by

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the pdf of $U = \ln Y$.
(b) Derive the pdf of $V = e^Y$.
(c) Derive the pdf of $W = F_Y(Y)$, where $F_Y(y)$ is the cdf of Y. Make sure to note the support of U, V and W.

13. Suppose that Y_1 and Y_2 are numbers generated independently and at random from the interval $(0, 1)$ so that the joint probability density function (pdf) of Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 1, & 0 < y_1 < 1, 0 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Define $U = Y_1Y_2$, the product of Y_1 and Y_2. Derive the pdf of U.

14. A triathlon is an athletic event made up of three contests (swimming, cycling, and running). Participants are timed for each contest, and the sum of the contest times represents the total time for the participant. For a certain triathlon, where time is measured in minutes, let

$$\begin{aligned} Y_1 &= \text{time for event 1} \\ Y_2 &= \text{time for event 2} \\ Y_3 &= \text{time for event 3.} \end{aligned}$$

Suppose Y_1, Y_2, and Y_3 are mutually independent.
(a) Suppose $Y_1 \sim \mathcal{N}(120, 100)$, $Y_2 \sim \mathcal{N}(70, 50)$, and $Y_3 \sim \mathcal{N}(30, 20)$. What is the distribution of $U = Y_1 + Y_2 + Y_3$? Find the mean and variance of U.
(b) What is the probability a participant in this triathlon will take longer than 250 minutes to complete all three events?

15. A machine produces spherical containers whose radii, Y, vary according to the probability density function (pdf) given by

$$f_Y(y) = \begin{cases} 3y^2, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the pdf of U, the surface area of the containers. Recall the surface area of a sphere is given by $U = h(Y) = 4\pi Y^2$.
(b) Find an expression for $E(U^k)$, where $k > 0$, and evaluate your expression at $k = 1$.

16. Suppose Y_1, Y_2 is an iid sample of size $n = 2$ from a standard normal distribution. Recall the standard normal pdf is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, & -\infty < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Define the random variables $U_1 = h_1(Y_1, Y_2)$ and $U_2 = h_2(Y_1, Y_2)$, where

$$\begin{aligned} h_1(Y_1, Y_2) &= Y_1 + Y_2 \\ h_2(Y_1, Y_2) &= Y_1 - Y_2. \end{aligned}$$

(a) Use a bivariate transformation to derive the joint distribution of U_1 and U_2.
(b) What is the (marginal) distribution of U_2?

17. A continuous random variable Y is said to have a (standard) logistic distribution if its probability density function (pdf) is given by

$$f_Y(y) = \begin{cases} e^{-y}(1 + e^{-y})^{-2}, & -\infty < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Prove that

$$U = h(Y) = \frac{1}{1 + e^{-Y}}$$

has a $\mathcal{U}(0, 1)$ distribution.
(b) Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from $f_Y(y)$. Find the pdf of $Y_{(n)}$.

18. Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from a shifted-exponential distribution with probability density function (pdf)

$$f_Y(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{otherwise.} \end{cases}$$

Derive the pdf of the minimum order statistic $Y_{(1)}$ and find $E(Y_{(1)})$.

19. Suppose $Y \sim \mathcal{U}(-1, 1)$. Find the pdf of $U = Y^2$. Make sure to state the support of U.

20. Suppose Y follows a Pareto distribution with cumulative distribution function

$$F_Y(y) = \begin{cases} 0, & y < \beta \\ 1 - \left(\dfrac{\beta}{y}\right)^{\alpha}, & y \geq \beta, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. Note that I am giving you the cdf here (not the pdf). Find the probability density function of $U = 1/Y$ and find $E(U)$.

21. Suppose $Y \sim$ exponential(β), where $\beta > 0$. Derive the moment generating function (mgf) of $U = 2Y/\beta$, making certain to note the values at which this mgf is defined.

22. Suppose Y_1, Y_2, Y_3 are mutually independent random variables, each with common pdf

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$U_1 = \frac{Y_1}{Y_1 + Y_2}, \qquad U_2 = \frac{Y_1 + Y_2}{Y_1 + Y_2 + Y_3}, \qquad \text{and} \qquad U_3 = Y_1 + Y_2 + Y_3.$$

Show U_1, U_2, and U_3 are mutually independent and determine each marginal distribution.

23. Suppose $\mathbf{Y} = (Y_1, Y_2, Y_3)$ is a trivariate random vector with joint pdf

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \begin{cases} 6e^{-y_1-y_2-y_3}, & 0 < y_1 < y_2 < y_3 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Define the random variables

$$U_1 = Y_1, \qquad U_2 = Y_2 - Y_1, \qquad \text{and} \qquad U_3 = Y_3 - Y_2.$$

(a) Derive the joint pdf of $\mathbf{U} = (U_1, U_2, U_3)$. Identify the marginal distributions of U_1, U_2, and U_3 separately.
(b) Find the mean and variance of $W = -2U_1 + 3U_2 - U_3$.

24. Suppose X_1, X_2 are independent random variables where $X_1 \sim \chi^2(r_1)$ and $X_2 \sim \chi^2(r_2)$.
(a) Under what condition are X_1, X_2 iid?
(b) Derive the joint probability density function (pdf) of

$$U_1 = \frac{X_1}{X_2} \quad \text{and} \quad U_2 = X_1 + X_2.$$

(c) Show that U_1 and U_2 are independent and that $U_2 \sim \chi^2(r_1 + r_2)$.

25. Tomorrow a sum of \$10,000 will be invested at rate R. Tomorrow's rate is unknown at the moment, so suppose R has a uniform distribution from 0.02 to 0.04; i.e., the probability density function (pdf) of R is

$$f_R(r) = \begin{cases} 50, & 0.02 \leq r \leq 0.04 \\ 0, & \text{otherwise.} \end{cases}$$

The \$10,000 will be compounded instantaneously for one year so

$$U = h(R) = 10000e^R$$

represents the amount at the end of the year.
(a) Find the pdf of U. Make sure to note the support.
(b) Find $E(U)$, the expected amount of the investment after one year.

26. Suppose Y_1, Y_2, Y_3, Y_4 are mutually independent and identically distributed (iid) exponential random variables, each with common pdf

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the moment generating function of the sum $T = Y_1 + Y_2 + Y_3 + Y_4$. What is the distribution of T?
(b) Find the pdf of the maximum order statistic $Y_{(4)}$.

27. Suppose Y_1 and Y_2 are random variables with joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 < y_1 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Show that

$$U = \frac{Y_1}{Y_2} \sim \mathcal{U}(0, 1).$$

That is, U follows a uniform distribution from 0 to 1.

28. Suppose Y is a standard normal random variable; i.e., the probability density function (pdf) of Y is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, & -\infty < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Define

$$U = |Y|,$$

the absolute value of Y. Use the cumulative distribution function (cdf) technique to show the pdf of U is

$$f_U(u) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-u^2/2}, & u > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Now define
$$V = U^2.$$
Prove that V has gamma distribution with shape parameter $\alpha = 1/2$ and scale parameter $\beta = 2$. What is another name for this distribution?

29. An actuary models the annual demand for two insurance products Y_1 and Y_2 using the joint probability density function (pdf)
$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 2e^{-(y_1+y_2)}, & 0 < y_2 < y_1 < \infty \\ 0, & \text{otherwise.} \end{cases}$$
Use the cumulative distribution function (cdf) technique to derive the pdf of
$$U = h(Y_1, Y_2) = Y_1 + Y_2,$$
the total demand of the two products combined.

30. A machine produces spherical containers of different sizes. The radius R of a container produced from the machine is modeled using the probability density function (pdf)
$$f_R(r) = \begin{cases} 5r^4, & 0 < r < 1 \\ 0, & \text{otherwise.} \end{cases}$$
Suppose $R_1, R_2, ..., R_{10}$ are the radii measured on an iid sample of $n = 10$ containers; i.e., $R_1, R_2, ..., R_{10}$ are iid from $f_R(r)$.
(a) Determine the pdf of the minimum radius $R_{(1)}$. Make sure to note the support.
(b) Graph the pdf of R_1 and the pdf of $R_{(1)}$ side by side.
(c) Calculate $P(R_{(1)} > 0.8)$ and compare with $P(R_1 > 0.8)$. Explain why these answers are different.

31. In a clinical trial that compares two treatments, let
$$\begin{aligned} Y_1 &= \text{the proportion of patients which respond to Treatment I} \\ Y_2 &= \text{the proportion of patients which respond to Treatment II.} \end{aligned}$$
Suppose the joint probability density function (pdf) of Y_1 and Y_2 is given by
$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 6(1-y_1)y_2^2, & 0 < y_1 < 1\ , 0 < y_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$
(a) Use a bivariate transformation to find the joint pdf of
$$U_1 = \frac{Y_1}{Y_2} \quad \text{and} \quad U_2 = Y_1.$$
Make sure to note the support.
(b) Find the marginal pdf of U_1 and calculate $P(0.9 < U_1 < 1.1)$.

32. Suppose $Y_1, Y_2, ..., Y_n$ are independent and identically distributed (iid) random variables, each having an exponential distribution with mean $\beta = 1$.
(a) Derive the moment generating function (mgf) of

$$U = 2\sum_{i=1}^{n} Y_i.$$

What is the distribution of U?
(b) Let $Y_{(1)}$ denote the minimum order statistic. Determine the distribution of

$$V = 2nY_{(1)}.$$

(c) Let $Y_{(n)}$ denote the maximum order statistic and define

$$W_n = Y_{(n)} - \ln n.$$

Show the cumulative distribution function (cdf) of W_n is

$$F_{W_n}(w) = \left(1 - \frac{e^{-w}}{n}\right)^n,$$

for $w \in \mathbb{R}$. What function does $F_{W_n}(w)$ converge to as $n \to \infty$?

33. Suppose Y is a uniform random variable from 0 to 1, that is, $Y \sim \mathcal{U}(0,1)$, so that the probability density function (pdf) of Y is

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Define the function

$$U = h(Y) = \frac{Y}{1+Y}.$$

(a) Using the method of distribution functions or the method of transformations (you pick), derive the pdf of U. Make sure to note the support of U when you write out your density.
(b) With your pdf in part (a), calculate $E(U)$. Then calculate

$$E\left(\frac{Y}{1+Y}\right)$$

by using the distribution of Y. Why are these expectations equal?

34. A random variable Y is said to have a logistic distribution if its cumulative distribution function (cdf) is given by

$$F_Y(y) = \frac{1}{1+e^{-y}}, \quad -\infty < y < \infty.$$

(a) Derive the probability density function (pdf) of Y and graph it.
(b) Show that Y and $-Y$ have the same distribution. *Hint:* Show the cdf of

$$U = h(Y) = -Y$$

is the same as the cdf of Y. If two random variables have the same cdf, then they have the same distribution.
(c) Derive the pdf of

$$V = g(Y) = 1 + e^Y$$

and show that $E(V)$ does not exist.

35. An electrical engineer uses the probability density function (pdf)

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 2e^{-(y_1+y_2)}, & 0 < y_1 < y_2 < \infty \\ 0, & \text{otherwise} \end{cases}$$

to model the lifetime of two components, denoted by Y_1 and Y_2, respectively (measured in years). It is known the lifetime of the second component Y_2 always exceeds that of the first Y_1.
(a) Derive the pdf of

$$U = Y_2 - Y_1,$$

the excess lifetime of the second component when compared to the first. Use either the method of distribution functions or a bivariate transformation.
(b) Calculate $E(U)$, the mean excess lifetime of the second component.

36. Suppose Y_1, Y_2, Y_3, Y_4, Y_5 are mutually independent $\mathcal{N}(0, 1)$ random variables.
(a) Show the moment generating function of

$$R = Y_1 + Y_2 + Y_3 + Y_4 + Y_5$$

is

$$m_R(t) = \exp\left(\frac{5t^2}{2}\right).$$

What is the distribution of R?
(b) Show the moment generating function of

$$U = Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2$$

is

$$m_U(t) = \left(\frac{1}{1-2t}\right)^{5/2}, \quad t < 1/2.$$

What is the distribution of U?
(c) Argue that

$$V = \frac{Y_1^2 + Y_2^2}{Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2}$$

has a beta(α, β) distribution and determine the parameters α and β.
(d) Determine the distribution of

$$W = \frac{(Y_1 + Y_2)/\sqrt{2}}{(Y_3 + Y_4 + Y_5)/\sqrt{3}}.$$

37. I have decided to sell my house. I am willing to accept the maximum of four independent bids Y_1, Y_2, Y_3, Y_4 (in \$100,000s) as a selling price, where these random variables have a common probability density function (pdf)

$$f_Y(y) = \begin{cases} \dfrac{3y^2}{125}, & 0 < y < 5 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the distribution of the maximum bid $Y_{(4)}$. This distribution describes (probabilistically) the selling price of my house.
(b) Find the mean and variance of $Y_{(4)}$.
(c) Find the joint pdf of the minimum bid $Y_{(1)}$ and the maximum bid $Y_{(4)}$.
(d) Calculate the covariance of $Y_{(1)}$ and $Y_{(4)}$. If you cannot complete the covariance calculation by hand, just show as many details as possible (the integration here is awful).