1. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Bernoulli(p) population distribution, where $0 < p < 1$ is unknown. The population pmf is

$$p_Y(y|p) = \begin{cases} p^y(1-p)^{1-y}, & y = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Prove that $\overline{Y}$ is the maximum likelihood estimator of p.
(b) Find the maximum likelihood estimator of $\tau(p) = \log[p/(1-p)]$, the *log-odds* of p.

2. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Weibull population distribution with $m = 3$ and $\alpha = 1/\theta$ so that the common pdf

$$f_Y(y|\theta) = \begin{cases} 3\theta y^2 e^{-\theta y^3}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the minimum variance unbiased estimator (MVUE) for $1/\theta$. *Hint:* Start by finding a sufficient statistic.

3. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Poisson(θ) distribution, where $\theta > 0$ is unknown.
(a) Argue $\overline{Y}$ is a consistent estimator for θ.
(b) Use the CLT to determine the large-sample (approximate) distribution of $\overline{Y}$.
(c) Find a consistent estimator of $g(\theta) = \ln\theta$.
(d) The third sample moment $m_3' = n^{-1}\sum_{i=1}^n Y_i^3$ converges in probability to what constant?

4. An engineering component has a lifetime Y which follows a shifted exponential distribution, in particular, the probability density function (pdf) of Y is

$$f_Y(y|\theta) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{otherwise.} \end{cases}$$

The unknown parameter $\theta > 0$ measures the magnitude of the shift. From an iid sample of component lifetimes $Y_1, Y_2, ..., Y_n$, we would like to estimate θ.
(a) Show the pdf of $Y_{(1)}$, the minimum order statistic, is

$$f_{Y_{(1)}}(y) = \begin{cases} ne^{-n(y-\theta)}, & y > \theta \\ 0, & \text{otherwise.} \end{cases}$$

(b) It follows that $Y_{(1)}$ is a sufficient statistic for θ (you do not need to prove this). Find the minimum variance unbiased estimator (MVUE) of θ.

5. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a beta population distribution with $\alpha = 1$ and $\beta > 0$ so that the common pdf is

$$f_Y(y|\beta) = \begin{cases} \beta(1-y)^{\beta-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the method of moments estimator (MOM) of β.
(b) Find the maximum likelihood estimator (MLE) of β.
(c) Find the MLE of the population variance.

6. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Poisson(θ) distribution, where $\theta > 0$ is unknown.
(a) Prove that $\overline{Y}$ is the maximum likelihood estimator (MLE) of θ and find its variance.
(b) We know that $\overline{Y}$ is an unbiased estimator of θ, but so is

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2.$$

Explain or prove why.
(c) It turns out that

$$V(S^2) = \frac{\theta}{n} + \frac{2\theta^2}{n-1}.$$

You do not have to prove this fact. Which estimator has smaller variance, $\overline{Y}$ or S^2? Are you surprised? Why or why not?
(d) Use the Rao-Blackwell Theorem to establish the surprising identity $E(S^2|\overline{Y}) = \overline{Y}$.

7. Consider a toxicology study with k groups of animals who are given a drug at distinct dose levels $d_1, d_2, ..., d_k$, respectively (these are fixed by the experimenter; not random). The animals are monitored for a reaction to the drug. In group i, let n_i (fixed) denote the total number of animals dosed, and let Y_i denote the number of animals that respond to the drug. The observations $Y_1, Y_2, ..., Y_k$ are treated as independent random variables, where $Y_i \sim b(n_i, p_i)$; $i = 1, 2, ..., k$, where p_i is the probability that an individual animal responds to dose d_i. Recall that the pmf of Y_i is given by

$$f_{Y_i}(y_i|p_i) = \binom{n_i}{y_i} p_i^{y_i}(1-p_i)^{n_i - y_i},$$

for $y_i = 0, 1, ..., n_i$. A standard assumption in such toxicology studies is that

$$\ln\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 d_i,$$

for $i = 1, 2, ..., k$, where β_0 and β_1 are real parameters (this is merely logistic regression using dose as a predictor).
(a) Use algebra to show that p_i, when written as a function of β_0, β_1, and d_i, satisfies

$$p_i = \frac{\exp(\beta_0 + \beta_1 d_i)}{1 + \exp(\beta_0 + \beta_1 d_i)}.$$

(b) Find a two-dimensional sufficient statistic for (β_0, β_1). *Hint:* First, substitute the expression for p_i from part (a) into $f_{Y_i}(y_i|p_i)$ and write the likelihood function in terms of β_0 and β_1. Recall that $Y_1, Y_2, ..., Y_k$ are independent, so

$$L(\beta_0, \beta_1 | y_1, y_2, ..., y_k) = \prod_{i=1}^{k} f_{Y_i}(y_i|p_i).$$

Now, simplify using algebraic properties of exponential functions and use the Factorization Theorem for the multiple parameter case.
(c) After the data are collected, suppose that a 95 percent confidence interval for β_1 is computed to be $(-0.84, 1.31)$. What can you say about the relationship between between dosing and response to the drug?

8. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from an exponential population distribution with mean $\beta > 0$, where β is unknown.
(a) Is $\overline{Y}$ a sufficient statistic for β? If so, prove it. If not, explain/prove why not.
(b) Derive (using mgfs) the distribution of $\overline{Y}$.
(c) Find a function of $\overline{Y}$ that converges in distribution to a standard normal distribution as $n \to \infty$. Explain why your answer is correct.
(d) Find two consistent estimators of $\sigma^2 = \beta^2$. Explain why your answers are correct.

9. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a $\mathcal{N}(\mu, \sigma^2)$ population distribution, where both μ and σ^2 are unknown. To estimate σ^2, we will use an estimator of the form $\widehat{\sigma}^2 = cS^2$, where S^2 is the usual sample variance; that is,

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2,$$

and $c > 0$ is a strictly positive constant. Prove that the value of c that minimizes $\text{MSE}(\widehat{\sigma}^2) = \text{MSE}(cS^2)$ is given by

$$c = \frac{n-1}{n+1}.$$

10. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Poisson population distribution with mean $\lambda > 0$. In class, we proved that $\overline{Y}$ is the maximum likelihood estimator of λ.
(a) Find the maximum likelihood estimator of $P(Y_1 \leq 1) = g(\lambda) = e^{-\lambda} + \lambda e^{-\lambda}$.
(b) Find a consistent estimator of λ.
(c) Find a consistent estimator of $g(\lambda)$.
(d) Use the Central Limit Theorem and Slutsky's Theorem to prove that, as $n \to \infty$,

$$Z_n = \frac{\overline{Y} - \lambda}{\sqrt{\frac{S^2}{n}}} \xrightarrow{d} \mathcal{N}(0,1).$$

11. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a beta distribution with parameters $\alpha > 0$ and $\beta = 1$, so that the common pdf is

$$f_Y(y|\alpha) = \begin{cases} \alpha y^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that

$$T = -\sum_{i=1}^{n} \ln Y_i$$

is a sufficient statistic for α.

(b) Derive the sampling distribution of T. *Hint:* First, use a transformation or distribution function technique to show that $W = -\ln Y$ has an exponential distribution with mean $1/\alpha$. Then, T is the sum of iid exponentials. Note that you can do this part even if you couldn't do part (a).

12. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a population with common pdf

$$f_Y(y|\theta) = \begin{cases} \frac{1}{\theta}e^{-(y-1)/\theta}, & y > 1 \\ 0, & \text{otherwise.} \end{cases}$$

where $\theta > 0$ is unknown.
(a) Show $\widehat{\theta} = \overline{Y} - 1$ is the minimum-variance unbiased estimator (MVUE) for θ. *Hint:* First show that

$$T = \sum_{i=1}^{n} Y_i.$$

is a sufficient statistic.
(b) Find the MVUE for $\tau(\theta) = \theta^2$. *Hint:* Calculate $E(\widehat{\theta^2})$ first.

13. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Weibull population distribution with common pdf

$$f_Y(y|\theta) = \begin{cases} \alpha_0 \theta y^{\alpha_0 - 1} \exp(-\theta y^{\alpha_0}), & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where α_0 is known and $\theta > 0$ is unknown.
(a) Find the method of moments estimator of θ.
(b) Find the maximum likelihood estimator of θ.

14. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a gamma(α, β) population distribution, where both $\alpha > 0$ and $\beta > 0$ are unknown. Define

$$T = \sum_{i=1}^{n} Y_i.$$

(a) Is T a sufficient statistic for $\boldsymbol{\theta} = (\alpha, \beta)$? If so, prove it. If not, explain/prove why not.
(b) Derive (using mgfs) the sampling distribution of T.
(c) Find a function of T that converges in distribution to a standard normal distribution as $n \to \infty$. Explain why your answer is correct.
(d) Find a function of T that converges in probability to $\mu = \alpha\beta$ as $n \to \infty$. Explain why your answer is correct.

15. The Pareto distribution is used in economics to model income distributions. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Pareto population distribution, where the common pdf is

$$f_Y(y|\theta) = \begin{cases} \theta \nu_0^{\theta} y^{-(\theta+1)}, & y > \nu_0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 1$ is unknown. In this population-level model, ν_0 represents the minimum income level (measured in \$1,000's). We will assume that $\nu_0 = 65$; i.e., ν_0 is a known value.
(a) Find the method of moments estimator of θ.
(b) Find the maximum likelihood estimator of θ.
(c) Compute your MOM and MLE estimates with the following income data:

```
87.10   68.55   123.48   77.77   110.88   100.54   98.78   84.32
```

16. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Bernoulli(p) population distribution, where $0 < p < 1$ is unknown. The population pmf is

$$p_Y(y|p) = \begin{cases} p^y(1-p)^{1-y}, & y = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $\overline{Y}$ is a sufficient statistic for p.
(b) Find the MVUE of $P(Y = 0) = 1 - p$.

17. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a population distribution with common pdf

$$f_Y(y|\mu) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\mu)^2}, & -\infty < y < \infty \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu \in \mathbb{R}$ is an unknown parameter.
(a) Show that $\overline{Y}$ is a sufficient statistic for μ.
(b) Show the MVUE of $\tau(\mu) = \mu^2$ is

$$\widehat{\mu} = \overline{Y}^2 - \frac{1}{n}.$$

18. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a population distribution with common pdf

$$f_Y(y|\theta) = \begin{cases} \dfrac{y}{\theta} e^{-y^2/2\theta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The population parameter $\theta > 0$ is unknown.
(a) Find a sufficient statistic for θ. Then find the MVUE.
(b) Find the MOM of θ.
(c) Find the MLE of θ.

19. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a lognormal population distribution; i.e., the population pdf is

$$f_Y(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma y} e^{-(\ln y - \mu)^2/2\sigma^2}, \ y > 0.$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Both parameters are unknown.
(a) Derive $E(Y)$ and $E(Y^2)$. Then find the method of moments estimators of μ and σ^2. *Hint:* Recall that if $U \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = e^U$ is lognormal with pdf given as above. Therefore, $E(Y) = E(e^U)$, which is the mgf of U when $t = 1$.
(b) Find the maximum likelihood estimators of μ and σ^2. Don't worry about verifying 2nd order conditions.

20. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a gamma population distribution with shape parameter $\alpha = 2$ and scale parameter θ, where $\theta > 0$ is unknown.
(a) Using the definition, show that

$$T = \sum_{i=1}^{n} Y_i$$

is a sufficient statistic. By "using the definition," I mean do not use the Factorization Theorem.
(b) Here are two unbiased estimators of $\tau(\theta) = \theta^2$:

$$\begin{aligned} \widehat{\tau}_1 &= \frac{S^2}{2} \\ \widehat{\tau}_2 &= \frac{1}{2}\left(\frac{n}{2n+1}\right)\overline{Y}^2. \end{aligned}$$

The relative efficiency of $\widehat{\tau}_1$ to $\widehat{\tau}_2$ is

$$\text{eff}(\widehat{\tau}_1 \text{ to } \widehat{\tau}_2) = \frac{V(\widehat{\tau}_2)}{V(\widehat{\tau}_1)}.$$

Is $\text{eff}(\widehat{\tau}_1 \text{ to } \widehat{\tau}_2) < 1$, $\text{eff}(\widehat{\tau}_1 \text{ to } \widehat{\tau}_2) = 1$, or $\text{eff}(\widehat{\tau}_1 \text{ to } \widehat{\tau}_2) > 1$? Explain.

21. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a $\mathcal{N}(0, \sigma^2)$ population distribution. That is, the population mean μ is known and is equal to 0. The population variance $\sigma^2 > 0$ is unknown.
(a) Show that $\overline{Y}$ is not a sufficient statistic. *Hint:* Show the conditional distribution of the sample $\mathbf{Y}$ given $\overline{Y}$ is *not* free of σ^2.
(b) Find a function of $\overline{Y}$ that is an unbiased estimator of σ^2. *Hint:* Start by considering $\overline{Y}^2$.
(c) Is your unbiased estimator in part (b) the MVUE for σ^2? Explain.

22. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a population with probability density function

$$f_Y(y|\theta) = \begin{cases} \theta\left(\dfrac{1}{y+1}\right)^{\theta+1}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The population parameter $\theta > 0$ is unknown.
(a) Use the Factorization Theorem to show

$$T = \prod_{i=1}^{n}\left(\frac{1}{Y_i+1}\right)$$

is a sufficient statistic.
(b) Use the transformation method (from Chapter 6) to show

$$U = h(Y) = -\ln\left(\frac{1}{Y+1}\right) \sim \text{exponential}\left(\frac{1}{\theta}\right).$$

and therefore prove $V = -\ln T \sim \text{gamma}(n, 1/\theta)$.
(c) Find the MVUE of θ. *Hint:* Find a function of V that is an unbiased estimator of θ.

23. Consider the population-level probability density function (pdf)

$$f_Y(y|\theta) = \begin{cases} \frac{1}{2}(1+\theta y), & -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The population parameter θ describes the decay distribution of electrons from muon decay when $Y = \cos(W)$, and W is the angle measured in an experiment. The parameter θ, which is related to polarization, is to be estimated using an iid sample $Y_1, Y_2, ..., Y_n$ from $f_Y(y|\theta)$. The parameter θ satisfies $-1 < \theta < 1$ and is unknown.
(a) Is there a scalar (i.e., one-dimensional) statistic T which is sufficient? If so, find it. If not, show why not.
(b) Find the method of moments (MOM) estimator of θ. Is the MOM estimator unbiased?
(c) Write out the likelihood function $L(\theta|\mathbf{y})$ and also the score equation that would be solved to find the maximum likelihood estimator (MLE). Recall the score equation is simply

$$\frac{\partial}{\partial\theta} \ln L(\theta|\mathbf{y}) \stackrel{\text{set}}{=} 0.$$

The score equation can not be solved analytically in this example, so you can not find a closed-form expression for the MLE.

24. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a $\mathcal{N}(\mu_0, \sigma^2)$ population distribution, where the population mean μ_0 is known and the population variance $\sigma^2 > 0$ is unknown.
(a) Show that

$$T = \sum_{i=1}^{n} (Y_i - \mu_0)^2$$

is a sufficient statistic for σ^2.
(b) Consider the two point estimators of σ^2:

$$\begin{aligned} \widehat{\sigma}_1^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 \\ \widehat{\sigma}_2^2 &= \frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu_0)^2. \end{aligned}$$

Argue that both are unbiased.
(c) The relative efficiency of $\widehat{\sigma}_1^2$ to $\widehat{\sigma}_2^2$ is

$$\text{eff}(\widehat{\sigma}_1^2 \text{ to } \widehat{\sigma}_2^2) = \frac{V(\widehat{\sigma}_2^2)}{V(\widehat{\sigma}_1^2)}.$$

Is $\text{eff}(\widehat{\sigma}_1^2 \text{ to } \widehat{\sigma}_2^2) < 1$, $\text{eff}(\widehat{\sigma}_1^2 \text{ to } \widehat{\sigma}_2^2) = 1$, or $\text{eff}(\widehat{\sigma}_1^2 \text{ to } \widehat{\sigma}_2^2) > 1$? Explain.

25. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from an exponential(β) population distribution, where $\beta > 0$ is unknown. Recall in class, we showed $\sum_{i=1}^{n} Y_i$ is a sufficient statistic.
(a) Show that $T = Y_{(1)}$ is *not* a sufficient statistic. Do this by showing

$$f_{\mathbf{Y}|Y_{(1)}}(\mathbf{y}|t) = \frac{f_{\mathbf{Y}}(\mathbf{y})}{f_{Y_{(1)}}(t)}$$

depends on β.

(b) Argue that $\widehat{\beta} = nY_{(1)}$ is an unbiased estimator of β.
(c) Define the point estimator $\widehat{\beta}^* = E(\widehat{\beta}|\sum_{i=1}^n Y_i)$. Explain why $\widehat{\beta}^*$ is unbiased. How do $V(\widehat{\beta})$ and $V(\widehat{\beta}^*)$ compare?

26. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a population with probability density function (pdf)

$$f_Y(y) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{y^2}{\theta^3} \exp\left(-\frac{y^2}{2\theta^2}\right), & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where the population parameter $\theta > 0$ is unknown. This is called the *Maxwell distribution.* It is used in the physical sciences to model characteristics of gases and molecules.
(a) Show that

$$E(Y) = \sqrt{\frac{8}{\pi}}\,\theta$$

and then find the method of moments (MOM) estimator of θ.
(b) Find a sufficient statistic for θ.
(c) Determine the UMVUE for θ^2. *Hint:* Find a function of a sufficient statistic that is an unbiased estimator of θ^2.

27. Suppose $Y_1, Y_2, ..., Y_n$ are mutually independent random variables satisfying

$$Y_i \sim \text{Poisson}(\theta x_i),$$

for $i = 1, 2, ..., n$. That is, Y_i is distributed as Poisson with mean equal to θx_i. The population-level parameter $\theta > 0$ is unknown. The x_i's are fixed constants which are observed.
(a) Why aren't the Y_i's identically distributed?
(b) Show the likelihood function is given by

$$L(\theta|\mathbf{y}) = c\theta^{y_+}\exp(-\theta x_+),$$

where $y_+ = \sum_{i=1}^n y_i$, $x_+ = \sum_{i=1}^n x_i$, and c is a constant free of θ.
(c) Show the maximum likelihood estimator of θ is

$$\widehat{\theta} = \frac{Y_+}{x_+}.$$

(d) The data below provide the number of doctor's office/hospital visits per household Y and the number of residents per household x for a random sample of $n = 10$ health insurance policies:

y	7	2	5	2	4	7	4	9	3	12
x	4	1	3	1	5	3	4	6	1	5

Under the Poisson model described above, calculate the maximum likelihood estimate of θ for these data. Also provide an estimate of the standard error of $\widehat{\theta}$.

28. A random sample of 100 men produced a total of 25 who favored a controversial local issue. An independent random sample of 100 women produced a total of 35 who favored the same issue. Suppose p_M is the population proportion of men who favor the issue and p_W is the corresponding population proportion for women. Both parameters are unknown. The underlying statistical model in this situation can be written as

- $X_1, X_2, ..., X_{100}$ is an iid sample from a Bernoulli(p_M) distribution
- $Y_1, Y_2, ..., Y_{100}$ is an iid sample from a Bernoulli (p_W) distribution,

where the two samples are independent (therefore, all 200 random variables are mutually independent). In both samples, a "success" (i.e., $X_i = 1$ or $Y_j = 1$) is understood to mean the individual favors the issue.
(a) Show the likelihood function of p_M and p_W is

$$L(p_M, p_W) = p_M^{\sum_{i=1}^{100} x_i}(1-p_M)^{100-\sum_{i=1}^{100} x_i}\; p_W^{\sum_{j=1}^{100} y_j}(1-p_W)^{100-\sum_{j=1}^{100} y_j},$$

and calculate the maximum likelihood estimates of p_M and p_W using the information in the first paragraph.
(b) A friend of yours believes men and women favor this issue equally and hence $p_M = p_W = p$, where p is the common proportion in the two populations (still unknown). Determine the MLE of p and calculate it with the information in the first paragraph.
Hint: In part (b), you could rewrite the likelihood function in part (a) in terms of the common p and then maximize it over $0 < p < 1$. Alternatively, you could maximize $L(p_M, p_W)$ in two dimensions subject to the constraint that $p_M = p_W$. This can be done by using Lagrange multipliers.

29. An engineering component has a lifetime Y which follows a shifted exponential distribution, in particular, the probability density function (pdf) of Y is

$$f_Y(y|\theta) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{otherwise.} \end{cases}$$

The unknown parameter $\theta > 0$ measures the magnitude of the shift. From an iid sample of component lifetimes $Y_1, Y_2, ..., Y_n$, we would like to estimate θ.
(a) Show the pdf of $Y_{(1)}$, the minimum order statistic, is

$$f_{Y_{(1)}}(y) = \begin{cases} ne^{-n(y-\theta)}, & y > \theta \\ 0, & \text{otherwise.} \end{cases}$$

(b) Show that $Y_{(1)}$ is a consistent estimator of θ.

30. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a beta(α, β) population distribution, where $\alpha = \beta = \theta$, and $\theta > 0$ is unknown. That is, the population probability density function (pdf) is

$$f_Y(y) = \begin{cases} \dfrac{\Gamma(2\theta)}{[\Gamma(\theta)]^2}\, y^{\theta-1}(1-y)^{\theta-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the method of moments (MOM) estimator of θ. *Hint:* $E(Y) = 1/2$ is free of θ, so equate second moments.
(b) Is the MOM estimator a consistent estimator of θ? Prove or disprove.

31. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Poisson population distribution with mean $\lambda > 0$. The Central Limit Theorem (Chapter 7) assures us that

$$\frac{\overline{Y} - \lambda}{\sqrt{\frac{\lambda}{n}}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{as } n \to \infty.$$

(a) Carefully argue that

$$\frac{\overline{Y} - \lambda}{\sqrt{\frac{\overline{Y}}{n}}} \quad \text{and} \quad \frac{\overline{Y} - \lambda}{\sqrt{\frac{S^2}{n}}}$$

also converge in distribution to $\mathcal{N}(0, 1)$, as $n \to \infty$. *Hint:* Argue that $\overline{Y}$ and S^2 are both consistent estimators of λ and then use Slutsky's Theorem.
(b) Derive large-sample $100(1 - \alpha)\%$ confidence intervals for λ using each quantity in part (a). Note that each one is a large-sample pivot.
(c) Suppose $n = 100$, $\lambda = 10$, and you calculate

$$\frac{\overline{Y} - 10}{\sqrt{\frac{\overline{Y}}{100}}}.$$

Between what two values would you expect this statistic to fall with probability approximately equal to 0.95? What might you conclude if it fell well outside this range?

32. Suppose $Y_1, Y_2, ..., Y_n$ is an iid sample from a Bernoulli(p) population distribution, where $0 < p < 1$. The population pmf is

$$p_Y(y|p) = \begin{cases} p^y(1-p)^{1-y}, & y = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Prove that

$$\widehat{p} = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

is the maximum likelihood estimator (MLE) of p.
(b) Use the large-sample properties of MLEs to show that

$$\widehat{p} \sim \mathcal{AN}\left(p, \ \frac{p(1-p)}{n}\right),$$

for large n. Note this is the same result you would get by applying the CLT directly.