GROUND RULES:

- This exam contains 5 questions. Each question is worth 20 points. This exam is worth 100 points. An extra credit problem is given (written by Hagen) that is worth 10 extra points.
- This is an open-book and open-notes exam. This means you can use anything on our course web site in addition to the textbook. You can also use anything on my course web site for STAT 511. Do not use anything else. This means no communication with other individuals (except me) and do not use any other books/web sites.
- Each question contains subparts. On each part, there is opportunity for partial credit, so show all of your work and explain all of your reasoning. **Translation:** No work/no explanation means no credit.
- If you use R to answer any part or to check your work, please include all code and output as attachments. Do not just write out the code you used.
- Print your name legibly at the top of this page. Write your solution to each problem on its own page. You can use the back of the page if needed; you can append extra pages if needed. Collect your pages when done and staple in upper left corner. Keep all pages/solutions in order.
- I prefer you write out your solutions "by hand." In the past, I have noticed students who type their solutions (in LaTeX or Word) often do not provide enough detail and/or do not explain their reasoning very well.
- Your solutions should be turned in to me no later than **Monday, February 19 at 9:40am**. Your solutions should be delivered to me in person.

HONOR PLEDGE FOR THIS EXAM:

After you have finished the exam, please read the following statement and sign your name below it.

> *I promise that I did not discuss any aspect of this exam with anyone other than the instructor, that I neither gave nor received any unauthorized assistance on this exam, and that the work presented herein is entirely my own.*

1. Suppose Z is a continuous random variable with probability density function (pdf) $f_Z(z)$, which is known. Define

$$Y = \sigma Z + \mu,$$

where $\sigma > 0$ and $-\infty < \mu < \infty$ are both constants (not random). Note that Y is a linear function of Z.

(a) If $Z \sim f_Z(z)$, prove the pdf of Y is

$$f_Y(y) = \frac{1}{\sigma} f_Z\left(\frac{y-\mu}{\sigma}\right).$$

Hint: Use a transformation.

(b) In this part only, suppose Z has a standard normal distribution; i.e., $Z \sim \mathcal{N}(0,1)$. We know the pdf of Z is

$$f_Z(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, & -\infty < z < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Derive the pdf of $Y = \sigma Z + \mu$ using the result in part (a). Make sure to note the support of Y. What is the distribution of Y?

(c) In this part only, suppose Z has an exponential distribution with mean 1; i.e., $Z \sim$ exponential(1). We know the pdf of Z is

$$f_Z(z) = \begin{cases} e^{-z}, & z > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Derive the pdf of $Y = \sigma Z + \mu$ using the result in part (a). Make sure to note the support of Y. Find the mean and variance of Y. *Hint:* You already know $E(Z) = V(Z) = 1$ for $Z \sim$ exponential(1).

Note: The pdf of Y in part (c) is called a *shifted-exponential distribution.*

2. A company provides earthquake insurance. The premium Y_1 is modeled using a gamma distribution with $\alpha = 2$ and $\beta = 5$. The claim Y_2 is modeled using a gamma distribution with $\alpha = 1$ and $\beta = 5$. If Y_1 and Y_2 are independent, find the probability density function of

$$U = \frac{Y_1}{Y_2}.$$

Do not perform a bivariate transformation. Use the cdf method. Here are the steps below. Do them in this order.

(a) Show

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \dfrac{1}{125}\, y_1 e^{-(y_1+y_2)/5}, & y_1 > 0, y_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Draw a picture to show the cdf of U equals

$$F_U(u) = \int_{y_1=0}^{\infty} \int_{y_2=\frac{y_1}{u}}^{\infty} \frac{1}{125}\, y_1 e^{-(y_1+y_2)/5} dy_2 dy_1,$$

for $u > 0$. If I cannot see this from your picture, you get zero points!

(c) Do the double integral above and show

$$F_U(u) = \begin{cases} 0, & u \leq 0 \\ \left(\dfrac{u}{u+1}\right)^2, & u > 0. \end{cases}$$

(d) Differentiate $F_U(u)$ to obtain $f_U(u)$. Make sure to note the support of U.

(e) After you do parts (a)-(d), show that $E(U)$ does not exist.

3. Suppose Z is a continuous random variable with moment generating function (mgf) $m_Z(t)$, which is known. Define

$$Y = \sigma Z + \mu,$$

where $\sigma > 0$ and $-\infty < \mu < \infty$ are both constants (not random). Note that Y is a linear function of Z.

(a) Prove that

$$m_Y(t) = e^{\mu t} m_Z(\sigma t).$$

(b) In Problem 1, part (b), you know

$$m_Z(t) = e^{t^2/2},$$

where $Z \sim \mathcal{N}(0, 1)$. Use the result in part (a) to derive the mgf of $Y = \sigma Z + \mu$. What is the distribution of Y and how do you know?

(c) In Problem 1, part (c), you know

$$m_Z(t) = \frac{1}{1-t}, \quad \text{for } t < 1.$$

where $Z \sim$ exponential(1). Use the result in part (a) to derive the mgf of $Y = \sigma Z + \mu$. Then, calculate $E(Y)$ and $V(Y)$ by using the mgf of Y. Your answers should match those in Problem 1.

Also, provide a review discussion (from STAT 511) why the mgf of Z in part (c) is only defined when $t < 1$.

4. Suppose Y_1, Y_2, Y_3, Y_4 are mutually independent and identically distributed Bernoulli(p) random variables, where $0 < p < 1$. Recall the Bernoulli(p) probability mass function (pmf) is

$$p_Y(y) = \begin{cases} p^y(1-p)^{1-y}, & y = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P(Y_1 = Y_2 = Y_3 = Y_4)$. *Hint:* Among the $2^4 = 16$ possible values of $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$, only two of them have all entries equal.

(b) Find $E(Y_1Y_2Y_3Y_4)$.

(c) Find the moment generating function (mgf) of $U = Y_1Y_2^2Y_3^3Y_4^4$. What is the distribution of U? *Hint:* What values can the random variable U have?

(d) Find the mgf of $V = Y_1 + Y_2 + Y_3 + Y_4$. What is the distribution of V?

(e) Find the mgf of

$$W = \frac{Y_1 + Y_2 + Y_3 + Y_4}{4}.$$

What is the pmf of W?

5. Suppose $Y_1, Y_2, ..., Y_5$ are mutually independent lug nut lifetimes. Assume each lug nut has probability density function (pdf)

$$f_Y(y) = \begin{cases} 3y^2, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The lug nuts are arranged as a five-component engineering system.

(a) If the lug nuts are arranged in a *series system*, then the lifetime of the system will be determined by

$$Y_{(1)} = \min\{Y_1, Y_2, ..., Y_5\},$$

that is, the system will fail when the first lug nut fails. Find the pdf of $Y_{(1)}$.

(b) If the lug nuts are arranged in a *parallel system*, then the lifetime of the system will be determined by

$$Y_{(5)} = \max\{Y_1, Y_2, ..., Y_5\},$$

that is, the system will fail when the last lug nut fails. Find the pdf of $Y_{(5)}$.

(c) Find the pdf of the sample median $Y_{(3)}$ and calculate $P(Y_{(3)} > 0.5)$.

6. Extra Credit. I have 3 random variables X, Y, and Z. I know

- $X \sim \chi^2(n-2)$
- $Y \sim \text{gamma}(1,2)$
- $Z \sim \text{gamma}(3,2)$
- X, Y, and Z are mutually independent.

(a) Find the distribution of $W = X + Y + Z$.

(b) Find n such that $V(5W) = 100$.