6.4. The amount of flour used is a random variable $Y \sim \text{exponential}(\beta = 4)$. The pdf of Y is shown below (left). The cost is described in terms of a function of Y; i.e., $U = h(Y) = 3Y + 1$.

(a) Let's use the cdf technique to find the distribution of U. First, note that $y > 0 \Longrightarrow u = 3y + 1 > 1$. Therefore, the support of U is $R_U = \{u : u > 1\}$. For $u > 1$, the cdf of U is

$$F_U(u) = P(U \leq u) = P(3Y + 1 \leq u) = P\left(Y \leq \frac{u-1}{3}\right) = F_Y\left(\frac{u-1}{3}\right).$$

Recall the cdf of $Y \sim \text{exponential}(\beta = 4)$ is

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y/4}, & y > 0. \end{cases}$$

Therefore, for $y > 0 \Longleftrightarrow u > 1$, the cdf of $U = h(Y) = 3Y + 1$ is

$$F_U(u) = F_Y\left(\frac{u-1}{3}\right) = 1 - e^{-\left(\frac{u-1}{3}\right)/4} = 1 - e^{-(u-1)/12}.$$

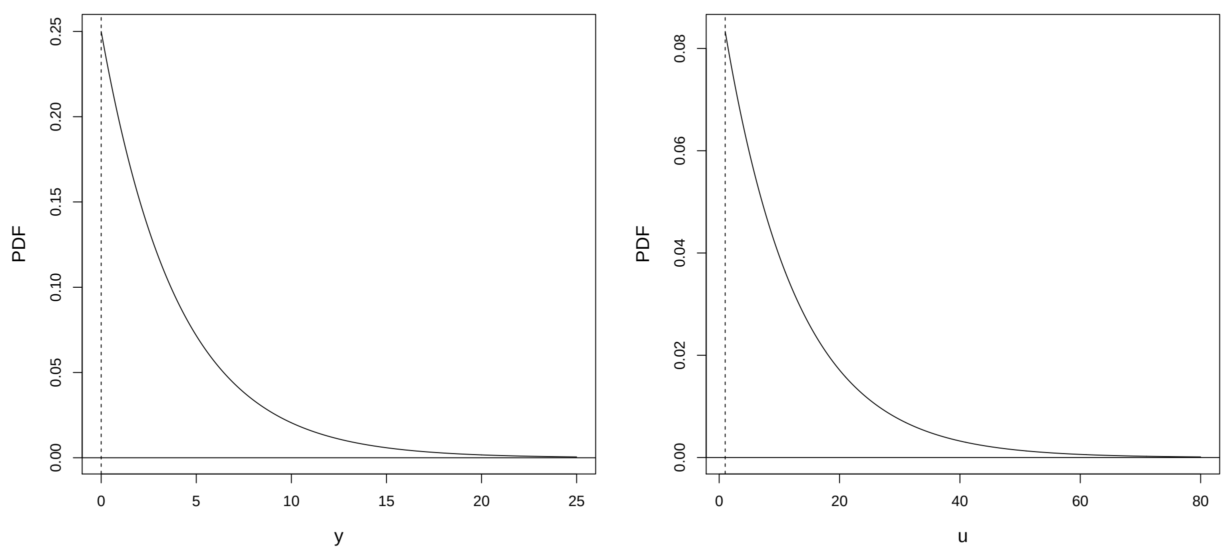
For $u > 1$, the pdf of U is

$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}\{1 - e^{-(u-1)/12}\} = \frac{1}{12}e^{-(u-1)/12}.$$

Summarizing,

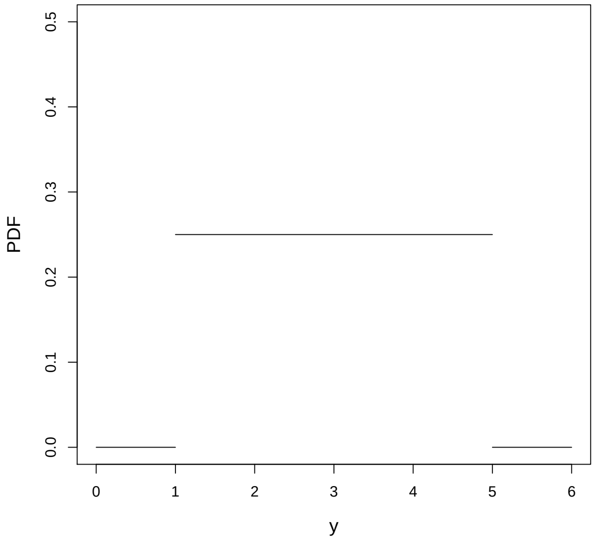
$$f_U(u) = \begin{cases} \dfrac{1}{12}e^{-(u-1)/12}, & u > 1 \\ 0, & \text{otherwise.} \end{cases}$$

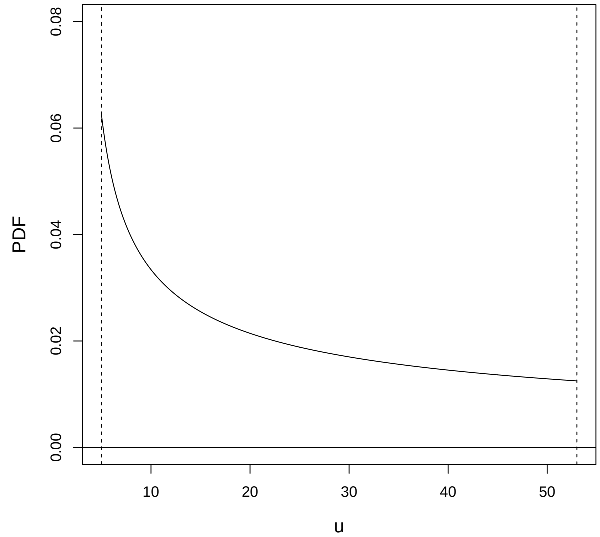
The pdf of U is shown below (right). Note that $f_U(u)$ is an exponential(12) pdf but with a horizontal shift of 1 unit to the right; i.e., a "shifted-exponential distribution."



(b) Using the pdf from part (a), the mean of U is

$$E(U) = \int_{\mathbb{R}} u f_U(u)du = \int_1^{\infty} \frac{u}{12}e^{-(u-1)/12}du.$$





In the last integral, let $v = u - 1$ so that $dv = du$. Therefore,

$$E(U) = \int_1^\infty \frac{u}{12} e^{-(u-1)/12} du = \int_0^\infty (v+1) \underbrace{\frac{1}{12} e^{-v/12}}_{\text{expo(12) pdf}} dv = E(V+1),$$

where $V \sim$ exponential(12). We have $E(U) = E(V+1) = E(V) + 1 = 12 + 1 = 13$. Of course, we would get the same answer by using the Law of the Unconscious Statistician; note that

$$E(U) = E(3Y+1) = 3E(Y) + 1 = 3(4) + 1 = 13.$$

6.5. The waiting time is a random variable $Y \sim \mathcal{U}(1,5)$. The pdf of Y is shown above (left). The cost is described in terms of a function of Y; i.e., $U = h(Y) = 2Y^2 + 3$.

Let's use the cdf technique to find the distribution of U. First, note that

$$1 < y < 5 \implies 5 < u < 53.$$

Therefore, the support of U is $R_U = \{u : 5 < u < 53\}$. For $5 < u < 53$, the cdf of U is

$$F_U(u) = P(U \le u) = P(2Y^2 + 3 \le u) = P\left(Y \le \sqrt{\frac{u-3}{2}}\right) = F_Y\left(\sqrt{\frac{u-3}{2}}\right).$$

Recall the cdf of $Y \sim \mathcal{U}(1,5)$ is given by

$$F_Y(y) = \begin{cases} 0, & y \le 1 \\ \dfrac{y-1}{4}, & 1 < y < 5 \\ 1, & y \ge 5. \end{cases}$$

Therefore, for $1 < y < 5 \iff 5 < u < 53$, the cdf of $U = h(Y) = 2Y^2 + 3$ is

$$F_U(u) = F_Y\left(\sqrt{\frac{u-3}{2}}\right) = \frac{\sqrt{\frac{u-3}{2}} - 1}{4}.$$

For $5 < u < 53$, the pdf of U is

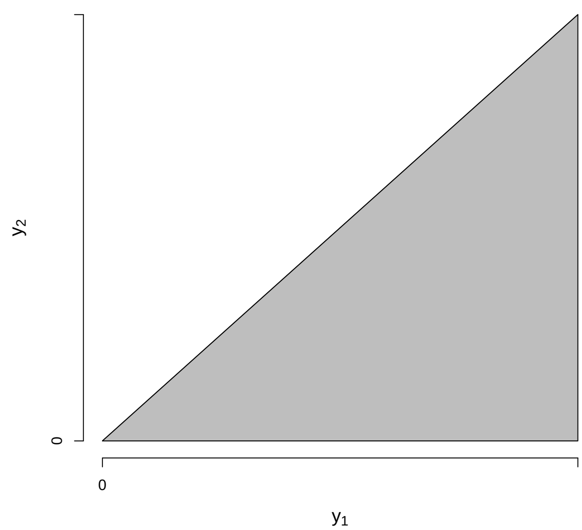
$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}\left(\frac{\sqrt{\frac{u-3}{2}} - 1}{4}\right) = \frac{1}{8}\left(\frac{u-3}{2}\right)^{-1/2}\left(\frac{1}{2}\right).$$

Summarizing,

$$f_U(u) = \begin{cases} \frac{1}{16}\left(\frac{u-3}{2}\right)^{-1/2}, & 5 < u < 53 \\ 0, & \text{otherwise.} \end{cases}$$

The pdf of U is shown at the top of the last page (right).

6.10. The support of (Y_1, Y_2) is the set $R = \{(y_1, y_2) : 0 \leq y_2 \leq y_1 < \infty\}$; this is the triangular region shown below. The upper boundary line is $y_2 = y_1$. The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value e^{-y_1} over this region and is otherwise equal to zero.

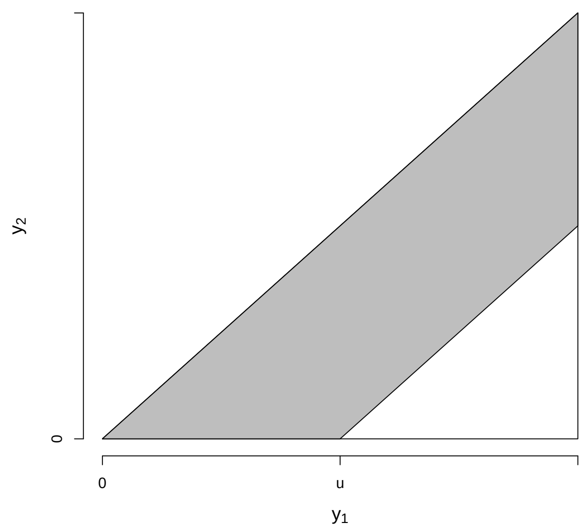


(a) We want to find the pdf of $U = Y_1 - Y_2$. We will use the cdf technique. First, observe that

$$y_1 \geq y_2 \geq 0 \implies u = h(y_1, y_2) = y_1 - y_2 \geq 0.$$

Therefore, the support of $U = h(Y_1, Y_2) = Y_1 - Y_2$ is $R_U = \{u : u \geq 0\}$. For $u \geq 0$, the cdf of U is

$$\begin{aligned} F_U(u) = P(U \leq u) &= P(Y_1 - Y_2 \leq u) \\ &= \int\int_{(y_1,y_2)\in B} f_{Y_1,Y_2}(y_1, y_2)dy_1dy_2 = \int\int_{(y_1,y_2)\in B} e^{-y_1}dy_1dy_2, \end{aligned}$$



where the set $B = \{(y_1, y_2) : y_1 \geq 0, y_2 \geq 0,\ y_1 - y_2 \leq u\}$. The region B is shown at the top of this page. Note that the boundary of B is

$$y_1 - y_2 = u \implies y_2 = y_1 - u,$$

a linear function of y_1 with slope 1 and intercept $-u$. The limits in the double integral (on the preceding page) come from this picture.

For $u \geq 0$, the cdf of U is

$$\begin{aligned} F_U(u) = P(U \leq u) &= \int_{y_2=0}^{\infty} \int_{y_1=y_2}^{y_2+u} e^{-y_1} dy_1 dy_2 \\ &= \int_{y_2=0}^{\infty} \left(-e^{-y_1}\right)\Big|_{y_1=y_2}^{y_2+u} dy_2 \\ &= \int_{y_2=0}^{\infty} [e^{-y_2} - e^{-(y_2+u)}] dy_2 \;=\; \underbrace{\int_{y_2=0}^{\infty} e^{-y_2} dy_2}_{=1} - \int_{y_2=0}^{\infty} e^{-(y_2+u)} dy_2. \end{aligned}$$

The first integral above is 1 because e^{-y_2} is the exponential(1) pdf and we are integrating it over $(0, \infty)$. The second integral is

$$\int_{y_2=0}^{\infty} e^{-(y_2+u)} dy_2 = -e^{-(y_2+u)}\Big|_{y_2=0}^{\infty} = 0 + e^{-u} = e^{-u}.$$

Therefore, for $u \geq 0$, the cdf of U is $F_U(u) = 1 - e^{-u}$. Summarizing,

$$F_U(u) = \begin{cases} 0, & u < 0 \\ 1 - e^{-u}, & u \geq 0. \end{cases}$$

We recognize this as an exponential(1) cdf; i.e., $U = Y_1 - Y_2 \sim$ exponential(1). For $u \geq 0$, the pdf of U is

$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}(1 - e^{-u}) = e^{-u}.$$

Summarizing,

$$f_U(u) = \begin{cases} e^{-u}, & u \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

This is the pdf of $U \sim$ exponential(1); i.e., an exponential pdf with mean $\beta = 1$.

(b) Based on our knowledge of the exponential distribution, we know

$$E(U) = 1 \quad \text{and} \quad V(U) = 1.$$

Comparing with Exercise 5.108, these are the same answers you would get if you calculated $E(U) = E(Y_1 - Y_2)$ and $V(U) = V(Y_1 - Y_2)$ by using the joint pdf of Y_1 and Y_2. To find $E(Y_1 - Y_2)$, we would calculate

$$E(Y_1 - Y_2) = \int\int_{(y_1,y_2)\in R} (y_1 - y_2) f_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 = \int_{y_1=0}^{\infty}\int_{y_2=y_1}^{\infty} (y_1 - y_2) e^{-y_1} dy_2 dy_1.$$

To get $V(Y_1 - Y_2)$, we could first calculate

$$E[(Y_1 - Y_2)^2] = \int\int_{(y_1,y_2)\in R} (y_1 - y_2)^2 f_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 = \int_{y_1=0}^{\infty}\int_{y_2=y_1}^{\infty} (y_1 - y_2)^2 e^{-y_1} dy_2 dy_1$$

and then use the variance computing formula

$$V(Y_1 - Y_2) = E[(Y_1 - Y_2)^2] - [E(Y_1 - Y_2)]^2.$$

We could also calculate $V(Y_1 - Y_2)$ by using

$$V(Y_1 - Y_2) = V(Y_1) + V(Y_2) - 2\text{Cov}(Y_1, Y_2).$$

As an exercise, try to calculate $E(U) = E(Y_1 - Y_2)$ and $V(U) = V(Y_1 - Y_2)$ by doing what is described above. It will be a lot of work, but it is a good review of Chapter 5 calculations. From the Law of the Unconscious Statistician, we know $E(Y_1 - Y_2) = 1$ and $V(Y_1 - Y_2) = 1$.

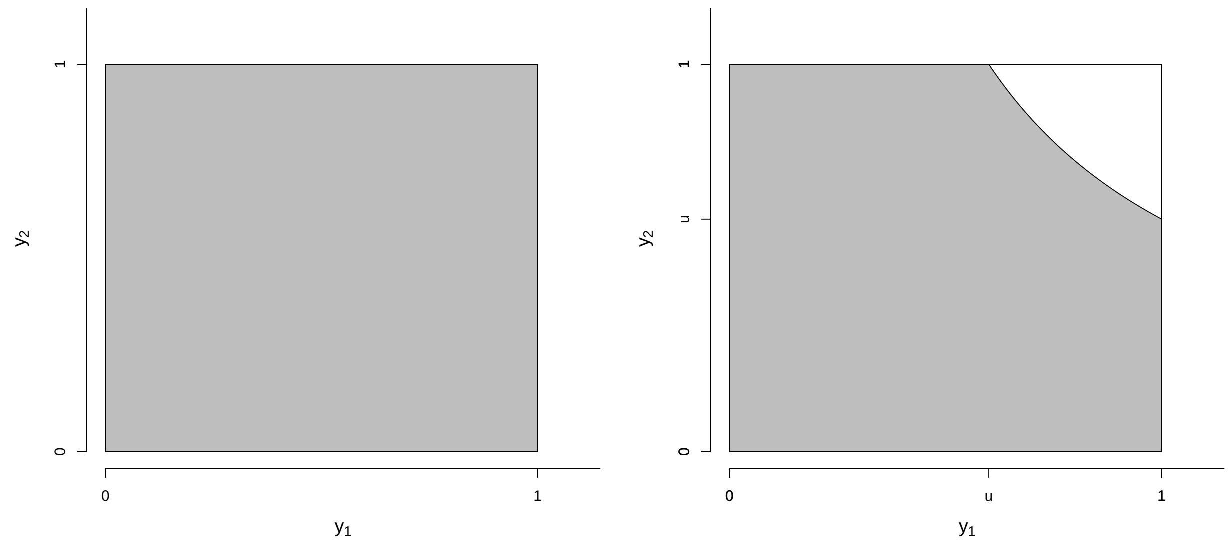
6.14. Because Y_1 and Y_2 are independent (by assumption), the joint pdf of Y_1 and Y_2, for $0 \leq y_1 \leq 1$ and $0 \leq y_2 \leq 1$, is given by

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) = 6y_1(1 - y_1) \times 3y_2^2 = 18y_1(1 - y_1)y_2^2.$$

Summarizing,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 18y_1(1 - y_1)y_2^2, & 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The support of (Y_1, Y_2) is the set $R = \{(y_1, y_2) : 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1\}$; i.e., the unit square. This region is shown at the top of the next page (left). The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $18y_1(1 - y_1)y_2^2$ over this region and is otherwise equal to zero.



We want to find the pdf of $U = Y_1Y_2$. We will use the cdf technique. First, observe that

$$0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1 \implies u = h(y_1, y_2) = y_1y_2 \in [0, 1].$$

Therefore, the support of $U = h(Y_1, Y_2) = Y_1Y_2$ is $R_U = \{u : 0 \leq u \leq 1\}$. For $0 \leq u \leq 1$, the cdf of U is

$$\begin{aligned} F_U(u) = P(U \leq u) &= P(Y_1Y_2 \leq u) \\ &= \iint_{(y_1,y_2)\in B} f_{Y_1,Y_2}(y_1, y_2)dy_1dy_2 = \iint_{(y_1,y_2)\in B} 18y_1(1-y_1)y_2^2dy_1dy_2, \end{aligned}$$

where the set $B = \{(y_1, y_2) : 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1,\ y_1y_2 \leq u\}$. The region B is shown at the top of this page (right). Note that the boundary of B is

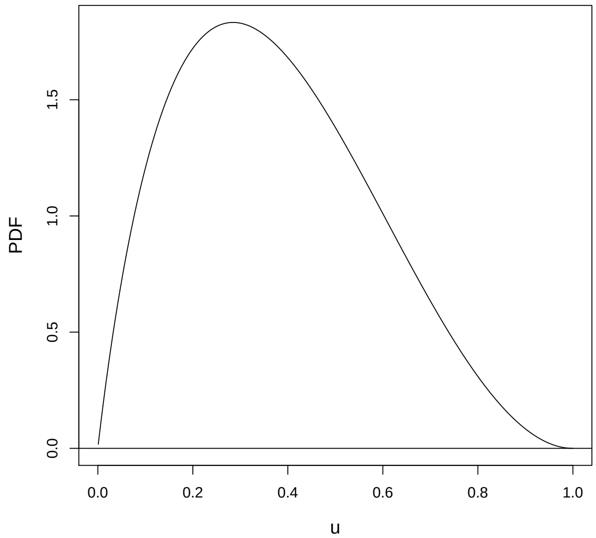
$$y_1y_2 = u \implies y_2 = \frac{u}{y_1}$$

a decreasing curvilinear function of y_1. The limits in the double integral above come from this picture.

Note: In this situation, it is much easier to integrate the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ over the complement of the shaded region above (right). That is,

$$F_U(u) = P(U \leq u) = P(Y_1Y_2 \leq u) = 1 - P(Y_1Y_2 > u).$$

The reason it is easier is that we can get the limits of integration easier (i.e., by integrating over the white region instead of the grey region). We calculate $P(Y_1Y_2 > u)$ by integrating the joint pdf over the white region.



For $0 \le u \le 1$, we have

$$\begin{aligned}
P(Y_1Y_2 > u) &= \int_{y_2=u}^{1}\int_{y_1=u/y_2}^{1} 18y_1(1-y_1)y_2^2 dy_1 dy_2 \\
&= \int_{y_2=u}^{1} 18y_2^2\left(\frac{y_1^2}{2}-\frac{y_1^3}{3}\right)\Bigg|_{y_1=u/y_2}^{1} dy_2 \\
&= \int_{y_2=u}^{1} 3y_2^2\left(3y_1^2-2y_1^3\right)\Big|_{y_1=u/y_2}^{1} dy_2 \\
&= \int_{y_2=u}^{1} 3y_2^2\left(1-\frac{3u^2}{y_2^2}+\frac{2u^3}{y_2^3}\right) dy_2 \\
&= \int_{y_2=u}^{1}\left(3y_2^2-9u^2+\frac{6u^3}{y_2}\right) dy_2 \\
&= \left(y_2^3-9u^2y_2+6u^3\ln y_2\right)\Big|_{y_2=u}^{1} \\
&= 1-9u^2+0-u^3+9u^3-6u^3\ln u \;=\; 1-9u^2+8u^3-6u^3\ln u.
\end{aligned}$$

Therefore, for $0 \le u \le 1$, we have

$$F_U(u) = 1 - P(Y_1Y_2 > u) = 1-(1-9u^2+8u^3-6u^3\ln u) = 9u^2-8u^3+6u^3\ln u.$$

Summarizing, the cdf of U is

$$F_U(u) = \begin{cases} 0, & u < 0 \\ 9u^2-8u^3+6u^3\ln u, & 0 \le u \le 1 \\ 1, & u > 1. \end{cases}$$

For $0 \leq u \leq 1$, the pdf of U is

$$\begin{aligned} f_U(u) = \frac{d}{du}F_U(u) &= \frac{d}{du}(9u^2 - 8u^3 + 6u^3 \ln u) \\ &= 18u - 24u^2 + (18u^2 \ln u + 6u^2) \\ &= 18u - 18u^2 + 18u^2 \ln u \;=\; 18u(1 - u + u \ln u). \end{aligned}$$

Summarizing,

$$f_U(u) = \begin{cases} 18u(1 - u + u \ln u), & 0 \leq u \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

A graph of this pdf appears at the top of the last page. I used R to ensure this pdf is valid; i.e., it integrates to 1.

```
> integrand <- function(u){18*u*(1-u+u*log(u))}
> integrate(integrand,lower=0,upper=1)
1 with absolute error < 3.7e-05
```

6.19. This exercise asks you establish a relationship between two new families of distributions; the power family and the Pareto family of distributions. Suppose $Y \sim \text{Pareto}(\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$. The cdf of Y is given in Exercise 6.18; it is

$$F_Y(y) = \begin{cases} 0, & y < \beta \\ 1 - \left(\dfrac{\beta}{y}\right)^{\alpha}, & y \geq \beta. \end{cases}$$

Consider the function

$$X = h(Y) = \frac{1}{Y}$$

Note that

$$y \geq \beta > 0 \implies 0 \leq \frac{1}{y} \leq \frac{1}{\beta}.$$

Therefore, the support of X is $R_X = \{x : 0 \leq x \leq 1/\beta\}$. For $0 \leq x \leq 1/\beta$, the cdf of X is

$$\begin{aligned} F_X(x) = P(X \leq x) = P\left(\frac{1}{Y} \leq x\right) &= P\left(Y > \frac{1}{x}\right) \\ &= 1 - P\left(Y \leq \frac{1}{x}\right) \\ &= 1 - F_Y\left(\frac{1}{x}\right) = 1 - \left[1 - \left(\frac{\beta}{1/x}\right)^{\alpha}\right] = (x\beta)^{\alpha}. \end{aligned}$$

Summarizing,

$$F_X(x) = \begin{cases} 0, & x < 0 \\ (x\beta)^{\alpha}, & 0 \leq x \leq 1/\beta \\ 1, & x > 1/\beta. \end{cases}$$

Letting $\theta = 1/\beta$, we have

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \left(\dfrac{x}{\theta}\right)^{\alpha}, & 0 \leq x \leq \theta \\ 1, & x > \theta. \end{cases}$$

That is, X follows a power family distribution with parameters α and $\theta = 1/\beta$.

6.20. Recall the pdf of $Y \sim \mathcal{U}(0,1)$ is

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) With $w = y^2$, note that $0 < y < 1 \implies 0 < w < 1$ as well. Therefore, the support of $W = Y^2$ is $R_W = \{0 < w < 1\}$. From Example 6.3 (notes), we derived a general expression for the pdf of $W = Y^2$. The pdf of W, where nonzero, is given by

$$f_W(w) = \frac{1}{2\sqrt{w}} \left[f_Y(\sqrt{w}) + f_Y(-\sqrt{w}) \right].$$

Note that for $0 < w < 1$,

$$\begin{aligned} f_Y(\sqrt{w}) &= 1 \\ f_Y(-\sqrt{w}) &= 0. \end{aligned}$$

Therefore, for $0 < w < 1$,

$$f_W(w) = \frac{1}{2\sqrt{w}}(1+0) = \frac{1}{2\sqrt{w}}.$$

Summarizing, the pdf of W is

$$f_W(w) = \begin{cases} \dfrac{1}{2\sqrt{w}}, & 0 < w < 1 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show this pdf is valid (i.e., it integrates to 1).

(b) We could use the cdf technique or the method of transformations to derive the pdf of $W = \sqrt{Y}$. I'll use the cdf technique. First, note that $0 < y < 1 \implies 0 < w < 1$ as well. Therefore, the support of W is $R_W = \{0 < w < 1\}$. The cdf of W, for $0 < w < 1$, is given by

$$F_W(w) = P(W \le w) = P(\sqrt{Y} \le w) = P(Y \le w^2) = F_Y(w^2),$$

where $F_Y(\cdot)$ is the cdf of $Y \sim \mathcal{U}(0,1)$. Recall that

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ y, & 0 < y < 1 \\ 1, & y \ge 1. \end{cases}$$

Therefore, for $0 < w < 1$, the cdf of W is

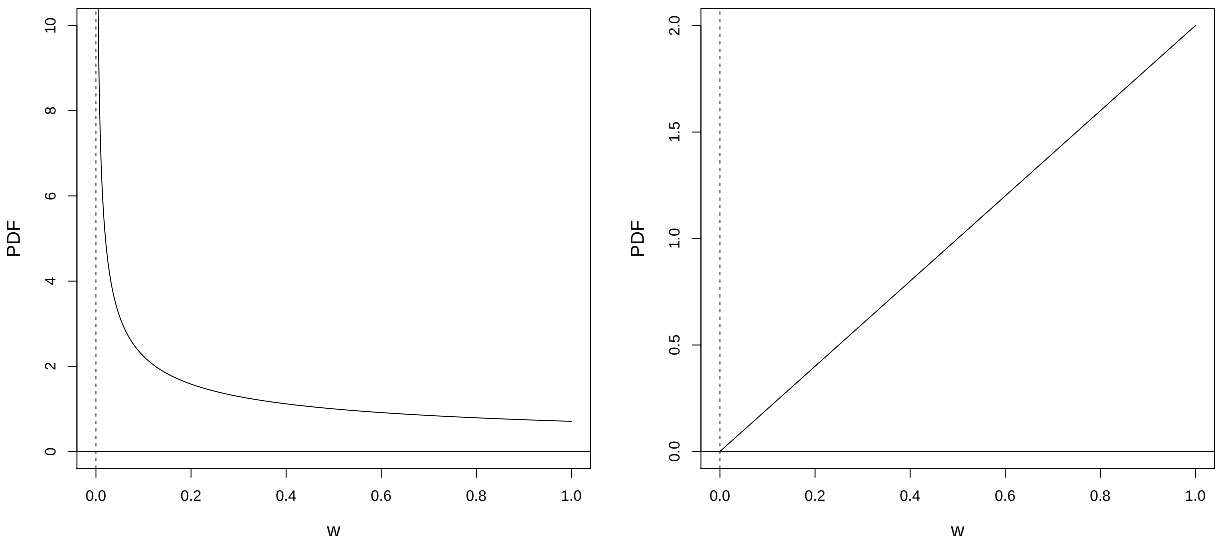
$$F_W(w) = F_Y(w^2) = w^2.$$

Summarizing,

$$F_W(w) = \begin{cases} 0, & w \le 0 \\ w^2, & 0 < w < 1 \\ 1, & w \ge 1. \end{cases}$$

We find the pdf of W by taking derivatives. For $0 < w < 1$, we have

$$f_W(w) = \frac{d}{dw} F_W(w) = \frac{d}{dw}\, w^2 = 2w.$$



Summarizing, the pdf of W is

$$f_W(w) = \begin{cases} 2w, & 0 < w < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that $W = \sqrt{Y}$ follows a beta distribution with $\alpha = 2$ and $\beta = 1$. Interesting!

Note: The pdf of $W = Y^2$ is shown above (left); the pdf of $W = \sqrt{Y}$ is shown above (right).

6.26. The pdf of $Y \sim \text{Weibull}(m, \alpha)$ is

$$f_Y(y) = \begin{cases} \dfrac{m}{\alpha} y^{m-1} e^{-y^m/\alpha}, & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $m > 0$ and $\alpha > 0$.

(a) To find the pdf of $U = h(Y) = Y^m$, we will use the transformation method. Note that

$$y > 0 \implies u = y^m > 0.$$

Therefore, the support of U is $R_U = \{u : u > 0\}$. In general, the function $u = h(y) = y^m$ is not 1:1 over $\mathbb{R} = (-\infty, \infty)$. However, it *is* 1:1 over $R_Y = \{y : y > 0\}$, the support of Y. Therefore, we can use the transformation method.

The inverse transformation is found as follows:

$$u = h(y) = y^m \implies y = u^{\frac{1}{m}} = h^{-1}(u).$$

Also, the derivative of the inverse transformation is

$$\frac{d}{du}h^{-1}(u) = \frac{d}{du}u^{\frac{1}{m}} = \frac{1}{m}u^{\frac{1}{m}-1}.$$

Therefore, for $u > 0$, the pdf of U is

$$\begin{aligned} f_U(u) &= f_Y(h^{-1}(u)) \left| \frac{d}{du}h^{-1}(u) \right| \\ &= \frac{m}{\alpha}(u^{\frac{1}{m}})^{m-1}e^{-(u^{\frac{1}{m}})^m/\alpha} \times \left| \frac{1}{m}u^{\frac{1}{m}-1} \right| = \frac{1}{\alpha}u^{1-\frac{1}{m}}e^{-u/\alpha}u^{\frac{1}{m}-1} = \frac{1}{\alpha}e^{-u/\alpha}. \end{aligned}$$

Summarizing, the pdf of $U = h(Y) = Y^m$ is

$$f_U(u) = \begin{cases} \frac{1}{\alpha}e^{-u/\alpha}, & u > 0 \\ 0, & \text{otherwise.} \end{cases}$$

We recognize this as an exponential pdf with mean α i.e., $U \sim \text{exponential}(\alpha)$.

(b) In this part, we want to derive a general expression for the kth moment of a Weibull random variable $Y \sim \text{Weibull}(m, \alpha)$. We can do this in two ways. First, we can calculate

$$E(Y^k) = \int_{\mathbb{R}} y^k f_Y(y)dy;$$

i.e., by using the distribution of Y. Second, we can note that

$$Y^k = (U^{\frac{1}{m}})^k = U^{\frac{k}{m}}$$

and calculate

$$E(Y^k) = E(U^{\frac{k}{m}}) = \int_{\mathbb{R}} u^{\frac{k}{m}} f_U(u)du;$$

i.e., by using the distribution of U. The Law of the Unconscious Statistician guarantees that both answers will be the same.

Let's do the second calculation; because $U \sim \text{exponential}(\alpha)$, we have

$$E(Y^k) = E(U^{\frac{k}{m}}) = \int_0^\infty u^{\frac{k}{m}} \times \frac{1}{\alpha}e^{-u/\alpha}du = \frac{1}{\alpha}\int_0^\infty u^{(\frac{k}{m}+1)-1}e^{-u/\alpha}du.$$

We recognize

$$u^{(\frac{k}{m}+1)-1}e^{-u/\alpha}$$

as a gamma kernel with shape parameter $\frac{k}{m} + 1$ and scale parameter α. Therefore, the last integral

$$\int_0^\infty u^{(\frac{k}{m}+1)-1}e^{-u/\alpha}du = \Gamma\left(\frac{k}{m}+1\right)\alpha^{\frac{k}{m}+1}.$$

Therefore, we have

$$E(Y^k) = E(U^{\frac{k}{m}}) = \frac{1}{\alpha}\Gamma\left(\frac{k}{m}+1\right)\alpha^{\frac{k}{m}+1} = \alpha^{\frac{k}{m}}\Gamma\left(\frac{k}{m}+1\right).$$

Note: When $k = 1$, we have the mean of Y; this is

$$E(Y) = \alpha^{\frac{1}{m}}\Gamma\left(\frac{1}{m}+1\right).$$

The second moment of Y arises when $k = 2$; we have

$$E(Y^2) = \alpha^{\frac{2}{m}}\Gamma\left(\frac{2}{m}+1\right).$$

The variance of $Y \sim \text{Weibull}(m, \alpha)$ is therefore

$$\begin{aligned} V(Y) = E(Y^2) - [E(Y)]^2 &= \alpha^{\frac{2}{m}}\Gamma\left(\frac{2}{m}+1\right) - \left[\alpha^{\frac{1}{m}}\Gamma\left(\frac{1}{m}+1\right)\right]^2 \\ &= \alpha^{\frac{2}{m}}\left\{\Gamma\left(\frac{2}{m}+1\right) - \left[\Gamma\left(\frac{1}{m}+1\right)\right]^2\right\}. \end{aligned}$$

These can be calculated numerically in R. For example, if $m = 2$ and $\alpha = 10$, then

```
> m = 2
> alpha = 10
> mean.Y = (alpha)^(1/m)*gamma((1/m)+1)
> var.Y = (alpha)^(2/m)*(gamma((2/m)+1)-(gamma((1/m)+1))^2)
> mean.Y
[1] 2.802496
> var.Y
[1] 2.146018
```

6.30. The pdf of $I \sim \mathcal{U}(9, 11)$ is

$$f_I(i) = \begin{cases} \frac{1}{2}, & 9 < i < 11 \\ 0, & \text{otherwise.} \end{cases}$$

To find the pdf of $P = h(I) = 2I^2$, we will use the transformation method. Note that

$$9 < i < 11 \implies 162 < p < 242.$$

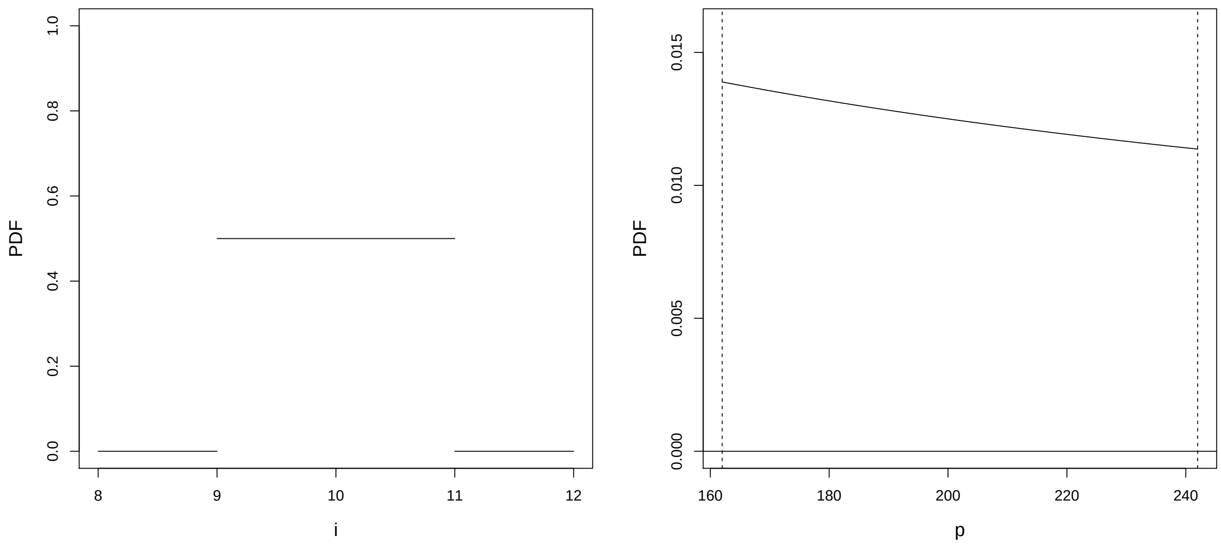
Therefore, the support of P is $R_P = \{p : 162 < p < 242\}$. In general, the function $p = h(i) = 2i^2$ is not 1:1 over $\mathbb{R} = (-\infty, \infty)$. However, it *is* 1:1 over $R_I = \{i : 9 < i < 11\}$, the support of I. Therefore, we can use the transformation method.

The inverse transformation is found as follows:

$$p = h(i) = 2i^2 \implies i = \sqrt{\frac{p}{2}} = h^{-1}(p).$$

The derivative of the inverse transformation is

$$\frac{d}{dp}h^{-1}(p) = \frac{d}{dp}\sqrt{\frac{p}{2}} = \frac{1}{2}\left(\frac{p}{2}\right)^{-1/2}\frac{1}{2} = \frac{1}{4}\sqrt{\frac{2}{p}}.$$



Therefore, for $162 < p < 242$, the pdf of P is

$$f_P(p) = f_I(h^{-1}(p)) \left| \frac{d}{dp} h^{-1}(p) \right| = \frac{1}{2} \times \left| \frac{1}{4}\sqrt{\frac{2}{p}} \right| = \frac{\sqrt{2}}{8} \frac{1}{\sqrt{p}}.$$

Summarizing, the pdf of $P = h(I) = 2I^2$ is

$$f_P(p) = \begin{cases} \frac{\sqrt{2}}{8} \frac{1}{\sqrt{p}}, & 162 < p < 242 \\ 0, & \text{otherwise.} \end{cases}$$

I used R to ensure this pdf is valid; i.e., it integrates to 1.

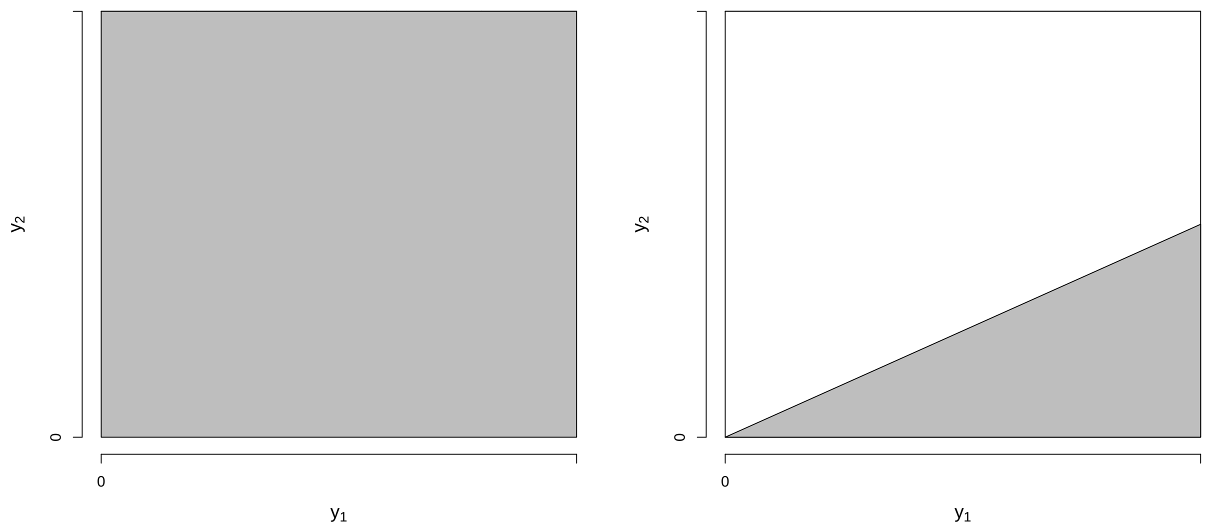
```
> integrand <- function(p){(sqrt(2)/8)*(1/sqrt(p))}
> integrate(integrand,lower=162,upper=242)
1 with absolute error < 1.1e-14
```

The pdf of $Y \sim \mathcal{U}(9, 11)$ and the pdf of $P \sim f_P(p)$ are shown above.

6.31. The support is $R = \{(y_1, y_2) : y_1 > 0, y_2 > 0\}$, the entire first quadrant. See the picture on the next page (left). The joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ is a three-dimensional function which takes the value $\frac{1}{8} y_1 e^{-(y_1+y_2)/2}$ over this region and is otherwise equal to zero.

To find the pdf of $U = Y_2/Y_1$, we will use the cdf technique. First, observe that

$$y_1 > 0, y_2 > 0 \implies u = h(y_1, y_2) = \frac{y_2}{y_1} > 0.$$



Therefore, the support of $U = h(Y_1, Y_2) = Y_2/Y_1$ is $R_U = \{u : u > 0\}$. For $u > 0$, the cdf of U is

$$F_U(u) = P(U \le u) = P\left(\frac{Y_2}{Y_1} \le u\right) = \underset{(y_1,y_2)\in B}{\int\int} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 dy_2,$$

where the set $B = \{(y_1, y_2) : y_1 > 0, y_2 > 0,\ \frac{y_2}{y_1} \le u\}$. The region B is shown at the top of this page (right). Note that the boundary of B is

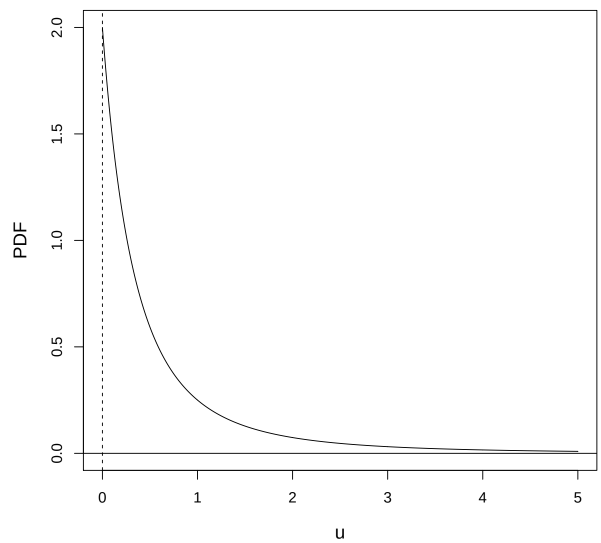
$$\frac{y_2}{y_1} = u \implies y_2 = uy_1,$$

a linear function of y_1 with intercept 0 and slope $u > 0$. The limits in the double integral above come from this picture.

For $u > 0$, the cdf of U is

$$\begin{aligned}
F_U(u) = P(U \le u) &= \int_{y_1=0}^{\infty} \int_{y_2=0}^{uy_1} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_2 dy_1 \\
&= \frac{1}{8} \int_{y_1=0}^{\infty} y_1 e^{-y_1/2} \left(\int_{y_2=0}^{uy_1} e^{-y/2} dy_2 \right) dy_1 \\
&= \frac{1}{8} \int_{y_1=0}^{\infty} y_1 e^{-y_1/2} \left(-2e^{-y/2} \Big|_0^{uy_2} \right) dy_1 \\
&= \frac{1}{4} \int_{y_1=0}^{\infty} y_1 e^{-y_1/2} \left(1 - e^{-uy_1/2} \right) dy_1 \\
&= \frac{1}{4} \left[\int_{y_1=0}^{\infty} y_1 e^{-y_1/2} dy_1 - \int_{y_1=0}^{\infty} y_1 e^{-y_1(1+u)/2} dy_1 \right].
\end{aligned}$$

Both integrals involve gamma kernels and are over $(0, \infty)$. The first integrand is $y_1 e^{-y_1/2}$, a gamma kernel with shape 2 and scale 2. The second integrand is $y_1 e^{-y_1(1+u)/2}$, a gamma kernel



with shape 2 and scale $2/(1+u)$. Therefore, for $u > 0$,

$$F_U(u) = P(U \le u) = \frac{1}{4}\left[\Gamma(2)2^2 - \Gamma(2)\left(\frac{2}{1+u}\right)^2\right] = 1 - \left(\frac{1}{1+u}\right)^2.$$

Summarizing,

$$F_U(u) = \begin{cases} 0, & u \le 0 \\ 1 - \left(\dfrac{1}{1+u}\right)^2, & u > 0. \end{cases}$$

For $u > 0$, the pdf of U is

$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}\left[1 - \left(\frac{1}{1+u}\right)^2\right] = \frac{2}{(1+u)^3}.$$

Summarizing,

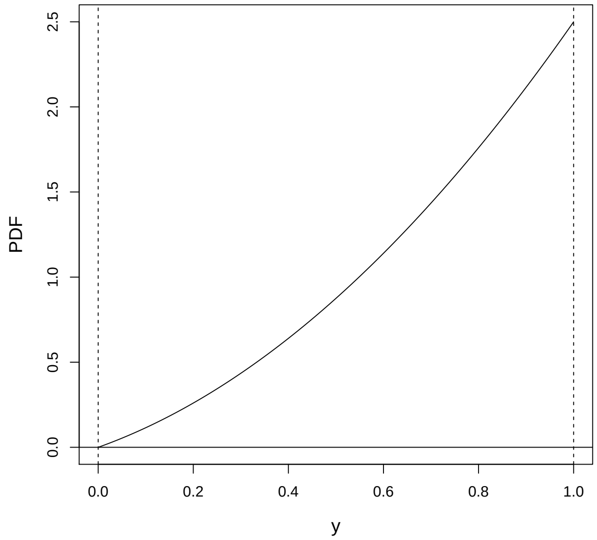
$$f_U(u) = \begin{cases} \dfrac{2}{(1+u)^3}, & u > 0 \\ 0, & \text{otherwise.} \end{cases}$$

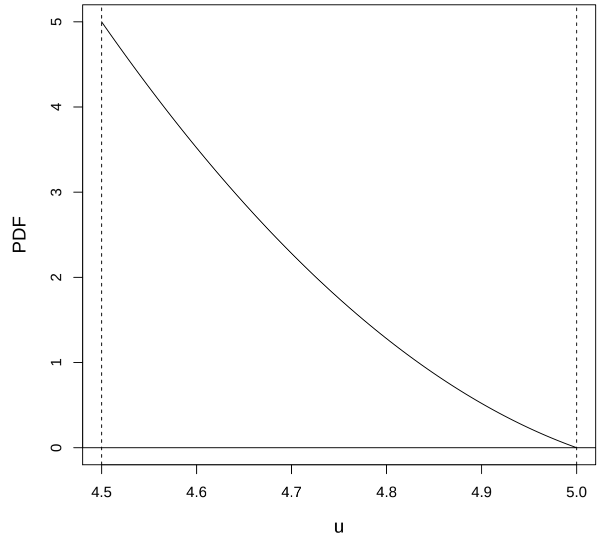
A graph of $f_U(u)$ is shown above. I used R to ensure this pdf is valid; i.e., it integrates to 1.

```
> integrand <- function(u){2/(1+u)^3}
> integrate(integrand,lower=0,upper=Inf)
1 with absolute error < 1.1e-14
```

6.33. The pdf of Y is

$$f_Y(y) = \begin{cases} \dfrac{3}{2}y^2 + y, & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$





To find the pdf of $U = h(Y) = 5 - Y/2$, we will use the transformation method. Note that

$$0 \leq y \leq 1 \implies 4.5 \leq u \leq 5.$$

Therefore, the support of U is $R_U = \{u : 4.5 \leq u \leq 5\}$. Note $u = h(y) = 5 - y/2$ is a linear function and hence is 1:1. Therefore, we can use the transformation method.

The inverse transformation is found as follows:

$$u = h(y) = 5 - y/2 \implies y = 2(5 - u) = h^{-1}(u).$$

The derivative of the inverse transformation is

$$\frac{d}{du}h^{-1}(u) = \frac{d}{du}[2(5-u)] = -2.$$

Therefore, for $4.5 \leq u \leq 5$, the pdf of U is

$$f_U(u) = f_Y(h^{-1}(u))\left|\frac{d}{du}h^{-1}(u)\right| = \left\{\frac{3}{2}[2(5-u)]^2 + 2(5-u)\right\} \times |-2| = 12(5-u)^2 + 4(5-u).$$

Summarizing, the pdf of $U = h(Y) = 5 - Y/2$ is

$$f_U(u) = \begin{cases} 12(5-u)^2 + 4(5-u), & 4.5 \leq u \leq 5 \\ 0, & \text{otherwise.} \end{cases}$$

I used R to ensure this pdf is valid; i.e., it integrates to 1.

```
> integrand <- function(u){12*(5-u)^2 + 4*(5-u)}
> integrate(integrand,lower=4.5,upper=5)
1 with absolute error < 1.1e-14
```

The pdf of Y and the pdf of $U = h(Y) = 5 - Y/2$ are shown above.