6.34. A Rayleigh random variable Y has pdf

$$f_Y(y) = \begin{cases} \dfrac{2y}{\theta} e^{-y^2/\theta}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that this pdf arises when

$$f_Y(y) = \begin{cases} \dfrac{m}{\theta} y^{m-1} e^{-y^m/\theta}, & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and $m = 2$. In other words, the Rayleigh(θ) distribution is a special case of the Weibull(m, θ) distribution with $m = 2$. We proved the general result

$$Y \sim \text{Weibull}(m, \theta) \implies U = h(Y) = Y^m \sim \text{exponential}(\theta)$$

in Exercise 6.26 (HW1) by using the transformation method. Therefore, arguing

$$Y \sim \text{Rayleigh}(\theta) \implies U = h(Y) = Y^2 \sim \text{exponential}(\theta)$$

is a special case of this general argument when $m = 2$. For fun, let's prove this result (when $m = 2$) by using the cdf technique and the mgf technique (in other words, all three methods "work" in this instance).

CDF technique: Let's first derive the cdf of $Y \sim$ Rayleigh(θ). When $y \leq 0$, the cdf

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{y} 0dt = 0.$$

For $y > 0$, the cdf

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \underbrace{\int_{-\infty}^{0} 0dt}_{=0} + \int_0^y \frac{2t}{\theta} e^{-t^2/\theta} dt = \int_0^y \frac{2t}{\theta} e^{-t^2/\theta} dt.$$

In the last integral, let

$$u = t^2 \implies du = 2t\ dt.$$

The limits on the integral change under this transformation. Note that

$$t : 0 \to y \implies u : 0 \to y^2.$$

Therefore, for $y > 0$,

$$\begin{aligned} F_Y(y) = \int_0^y \frac{2t}{\theta} e^{-t^2/\theta} dt &= \int_0^{y^2} \frac{2t}{\theta} e^{-u/\theta}\ \frac{du}{2t} \\ &= \int_0^{y^2} \frac{1}{\theta} e^{-u/\theta}\ du \\ &= \frac{1}{\theta}\left(-\theta e^{-u/\theta} \Big|_0^{y^2} \right) = e^{-u/\theta}\Big|_{y^2}^{0} = 1 - e^{-y^2/\theta}. \end{aligned}$$

Summarizing,

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y^2/\theta}, & y > 0. \end{cases}$$

We are now ready to derive the cdf of $U = Y^2$. For $u > 0$, it is

$$\begin{aligned} F_U(u) = P(U \le u) &= P(Y^2 \le u) \\ &= P(Y \le \sqrt{u}) \\ &= F_Y(\sqrt{u}) \;=\; 1 - e^{-(\sqrt{u})^2/\theta} \;=\; 1 - e^{-u/\theta}. \end{aligned}$$

Summarizing,

$$F_U(u) = \begin{cases} 0, & u \le 0 \\ 1 - e^{-u/\theta}, & u > 0. \end{cases}$$

We recognize this as the cdf of $U \sim \text{exponential}(\theta)$. Therefore, we are done.

MGF technique: We derive the mgf of $U = Y^2$ and show that it matches the mgf of an exponential random variable with mean θ. The mgf of U is

$$m_U(t) = E(e^{tU}) = E(e^{tY^2}) = \int_0^\infty e^{ty^2} \times \frac{2y}{\theta} e^{-y^2/\theta} dy \;=\; \int_0^\infty \frac{2y}{\theta} e^{ty^2 - y^2/\theta} dy.$$

In the exponent of $e^{ty^2 - y^2/\theta}$, write

$$ty^2 - \frac{y^2}{\theta} = -y^2\left(\frac{1}{\theta} - t\right) = -y^2 \Big/ \left(\frac{1}{\theta} - t\right)^{-1} = -y^2/\eta,$$

where $\eta = \left(\frac{1}{\theta} - t\right)^{-1}$. Therefore, the last integral becomes

$$m_U(t) = \int_0^\infty \frac{2y}{\theta} e^{ty^2 - y^2/\theta} dy = \int_0^\infty \frac{2y}{\theta} e^{-y^2/\eta} dy.$$

Now, let

$$u = y^2 \implies du = 2y\ dy.$$

The limits on the integral do not change under this transformation. Note that

$$y : 0 \to \infty \implies u : 0 \to \infty.$$

Therefore,

$$m_U(t) = \int_0^\infty \frac{2y}{\theta} e^{-u/\eta} \frac{du}{2y} = \int_0^\infty \frac{1}{\theta} e^{-u/\eta} du = \frac{1}{\theta}\left(-\eta e^{-u/\eta}\Big|_0^\infty\right) = \frac{\eta}{\theta}(1-0) = \frac{\eta}{\theta},$$

provided that

$$\eta > 0 \iff t < \frac{1}{\theta}.$$

Therefore, for $t < 1/\theta$, we have

$$m_U(t) = \frac{1}{\theta}\left(\frac{1}{\frac{1}{\theta} - t}\right) = \frac{1}{\theta}\left(\frac{\theta}{1 - \theta t}\right) = \frac{1}{1 - \theta t}.$$

We recognize this mgf as the mgf of an exponential random variable with mean θ. Because mgfs are unique, we know $U \sim \text{exponential}(\theta)$.

(b) In HW1, we derived the mean and variance of $Y \sim \text{Weibull}(m, \theta)$ to be

$$\begin{aligned} E(Y) &= \theta^{\frac{1}{m}} \Gamma\left(\frac{1}{m}+1\right) \\ V(Y) &= \theta^{\frac{2}{m}} \left\{ \Gamma\left(\frac{2}{m}+1\right) - \left[\Gamma\left(\frac{1}{m}+1\right)\right]^2 \right\}. \end{aligned}$$

Therefore, for $Y \sim \text{Raleigh}(\theta)$, put in $m = 2$ and we get

$$E(Y) = \theta^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) = \theta^{\frac{1}{2}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi\theta}}{2}.$$

and

$$\begin{aligned} V(Y) = \theta \left\{ \Gamma\left(\frac{2}{2}+1\right) - \left[\Gamma\left(\frac{1}{2}+1\right)\right]^2 \right\} &= \theta \left\{ \Gamma(2) - \left[\Gamma\left(\frac{3}{2}\right)\right]^2 \right\} \\ &= \theta \left\{ 1 - \left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]^2 \right\} = \theta\left(1 - \frac{\pi}{4}\right). \end{aligned}$$

6.40. We know that $Y \sim \mathcal{N}(0,1) \implies Y^2 \sim \chi^2(1)$. Therefore, Y_1^2 and Y_2^2 are independent random variables, both distributed as $\chi^2(1)$. Recall the $\chi^2(1)$ mgf is given by

$$m_{Y^2}(t) = \left(\frac{1}{1-2t}\right)^{1/2},$$

for $t < 1/2$. Therefore, the mgf of $U = Y_1^2 + Y_2^2$ is

$$m_U(t) = m_{Y_1^2}(t) m_{Y_2^2}(t) = \left(\frac{1}{1-2t}\right)^{1/2} \left(\frac{1}{1-2t}\right)^{1/2} = \left(\frac{1}{1-2t}\right)^{2/2}.$$

We recognize this mgf as the mgf of a χ^2 random variable with 2 degrees of freedom. Because mgfs are unique, we know $U = Y_1^2 + Y_2^2 \sim \chi^2(2)$; i.e., the degrees of freedom simply "add."

6.42. The weight capacity $Y_1 \sim \mathcal{N}(5000, 300^2)$. The load $Y_2 \sim \mathcal{N}(4000, 400^2)$. The elevator will be overloaded when $Y_1 < Y_2$; i.e., when $U = Y_1 - Y_2 < 0$. Therefore, we want to find $P(Y_1 < Y_2) = P(U < 0)$.

In Example 6.13 (notes), we proved that linear combinations of mutually independent normal random variables are normally distributed; i.e.,

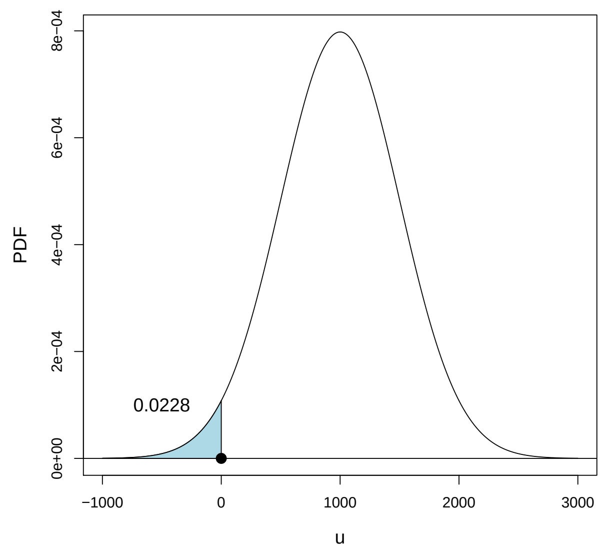
$$U = \sum_{i=1}^n a_i Y_i \sim \mathcal{N}\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right).$$

Note that

$$U = Y_1 - Y_2$$

is a special case of the linear combination above with $n = 2$, $a_1 = 1$, and $a_2 = -1$. Therefore, we know $U = Y_1 - Y_2$ is normally distributed with mean

$$a_1\mu_1 + a_2\mu_2 = 1(5000) + (-1)(4000) = 1000$$



and variance

$$a_1^2\sigma_1^2 + a_2^2\sigma_2^2 = 1^2(300^2) + (-1)^2(400^2) = 500^2.$$

That is, $U \sim \mathcal{N}(1000, 500^2)$. We can calculate $P(U < 0)$ in R; note that

```
> pnorm(0,1000,500)
[1] 0.02275013
```

Therefore,

$$P(Y_1 < Y_2) = P(U < 0) \approx 0.0228.$$

The pdf of $U \sim \mathcal{N}(1000, 500^2)$ is shown at the top of this page with the probability $P(U < 0)$ shown shaded.

6.45. We are given

$$\begin{aligned} Y_1 &= \text{amount of sand (in yards)} \sim \mathcal{N}(10, 0.5^2) \\ Y_2 &= \text{amount of cement (in 100s lbs)} \sim \mathcal{N}(4, 0.2^2). \end{aligned}$$

The total cost is

$$U = 100 + 7Y_1 + 3Y_2.$$

We are told to assume that Y_1 and Y_2 are independent. Under this assumption,

$$7Y_1 + 3Y_2$$

is a linear combination of independent normally distributed random variables with $n = 2$, $a_1 = 7$, and $a_2 = 3$. Therefore, it too is normally distributed with mean

$$a_1\mu_1 + a_2\mu_2 = 7(10) + 3(4) = 82$$

and variance

$$a_1^2\sigma_1^2 + a_2^2\sigma_2^2 = 7^2(0.5^2) + 3^2(0.2^2) = 12.61.$$

That is,

$$7Y_1 + 3Y_2 \sim \mathcal{N}(82, 12.61).$$

Now, the additive constant 100 merely shifts the $\mathcal{N}(82, 12.61)$ distribution 100 units to the right; therefore,

$$U = 100 + 7Y_1 + 3Y_2 \sim \mathcal{N}(182, 12.61).$$

Note: If you dislike the previous argument, you can derive the mgf of $U = 100 + 7Y_1 + 3Y_2$ directly and show that it matches the mgf of a $\mathcal{N}(182, 12.61)$ random variable. We do this now:

$$\begin{aligned} m_U(t) = E(e^{tU}) &= E[e^{t(100+7Y_1+3Y_2)}] \\ &= E(e^{100t}e^{7tY_1}e^{3tY_2}) \\ &\stackrel{Y_1 \perp\!\!\!\perp Y_2}{=} e^{100t}E(e^{7tY_1})E(e^{3tY_2}) \;=\; e^{100t}m_{Y_1}(7t)m_{Y_2}(3t), \end{aligned}$$

where $m_{Y_1}(t)$ is the $\mathcal{N}(10, 0.5^2)$ mgf and where $m_{Y_2}(t)$ is the $\mathcal{N}(4, 0.2^2)$ mgf. We have

$$m_{Y_1}(t) = \exp\left[10t + \frac{(0.5^2)t^2}{2}\right] \implies m_{Y_1}(7t) = \exp\left[70t + \frac{49(0.5^2)t^2}{2}\right]$$

and

$$m_{Y_2}(t) = \exp\left[4t + \frac{(0.2^2)t^2}{2}\right] \implies m_{Y_1}(3t) = \exp\left[12t + \frac{9(0.2^2)t^2}{2}\right]$$

Therefore,

$$\begin{aligned} m_U(t) = e^{100t}m_{Y_1}(7t)m_{Y_2}(3t) &= \exp(100t)\exp\left[70t + \frac{49(0.5^2)t^2}{2}\right]\exp\left[12t + \frac{9(0.2^2)t^2}{2}\right] \\ &= \exp\left\{182t + \frac{[49(0.5^2) + 9(0.2^2)]t^2}{2}\right\} \\ &= \exp\left(182t + \frac{12.61t^2}{2}\right). \end{aligned}$$

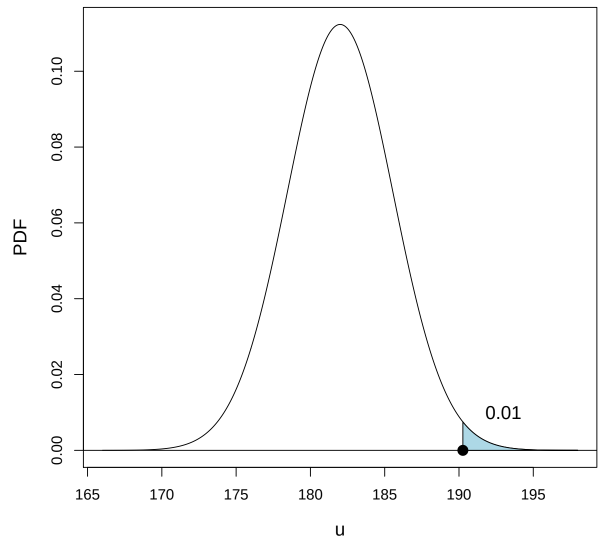
We recognize this as the mgf of a normal random variable with mean $\mu = 182$ and variance $\sigma^2 = 12.61$. Because mgfs are unique, we know that $U \sim \mathcal{N}(182, 12.61)$. Now, the bidding problem being asked is this. What should the manager bid on the job so that the total cost U will exceed the bid with probability 0.01? Let b denote the bid the manager makes. S/he wants to select b so that

$$P(U > b) = 0.01.$$

In other words, s/he wants to bid the 99th percentile ($p = 0.99$ quantile) of $U \sim \mathcal{N}(182, 12.61)$. In R, we have

```
> qnorm(0.99,182,sqrt(12.61))
[1] 190.261
```

Therefore, if s/he sets the bid at $b = 190.261$, then the total cost U will exceed this value with probability 0.01. See the figure at the top of the next page.



Remark: We are asked to comment on whether the amount of sand required and the amount of cement required for the construction job are independent; i.e., if it is reasonable to assume Y_1 and Y_2 are independent. On practical grounds, they probably aren't; in fact, we would expect them to be positively correlated (i.e., the more sand required for the construction job, the more cement will be required). Therefore, the solution we obtained ($b = 190.261$) isn't 100 percent correct if Y_1 and Y_2 are in fact correlated. However, we made the independence assumption so that we could get *a solution*. This is commonly done in statistical problems−we sometimes are forced to make simplifying assumptions so that we can get an answer. If we wanted to solve $P(U > b) = 0.01$ while allowing for dependence between Y_1 and Y_2, we would have to know the covariance of Y_1 and Y_2. If we knew this, then we could recalculate the distribution of U. It is still normal with mean $E(U) = 182$, but the variance would change as follows:

$$V(U) = V(100 + 7Y_1 + 3Y_2) = V(7Y_1 + 3Y_2) = 49V(Y_1) + 9V(Y_2) + 2(7)(3)\underbrace{\text{Cov}(Y_1, Y_2)}_{\text{would need this}}.$$

6.48. In this problem, we are given $Y_1 \sim \mathcal{N}(0, 1)$, $Y_2 \sim \mathcal{N}(0, 1)$, and Y_1 and Y_2 are independent. We want to find the distribution of

$$U = \sqrt{Y_1^2 + Y_2^2}.$$

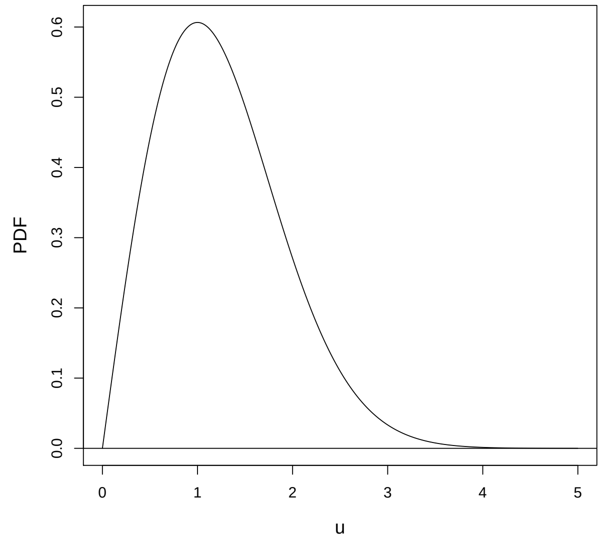
From Exercise 6.40, we already know

$$V = Y_1^2 + Y_2^2 \sim \chi^2(2).$$

Therefore, all we have to do is find the pdf of $U = h(V) = \sqrt{V}$, where $V \sim \chi^2(2) \overset{d}{=} \text{gamma}(1, 2)$. The pdf of V, for $v > 0$, is

$$f_V(v) = \frac{1}{\Gamma(1)2^1} v^{1-1} e^{-v/2} \;=\; \frac{1}{2} e^{-v/2},$$

which is the exponential(2) pdf with mean $\beta = 2$. In other words, the $\chi^2(2)$ pdf, the gamma$(1, 2)$ pdf, and the exponential(2) pdf are all the same pdf! Interesting!!



To find the pdf of $U = h(V) = \sqrt{V}$, we will use the transformation method. Note that

$$v > 0 \implies u = \sqrt{v} > 0.$$

Therefore, the support of U is $R_U = \{u : u > 0\}$. Also, the function $u = h(v) = \sqrt{v}$ is 1:1 over $R_V = \{v : v > 0\}$, the support of V. Therefore, we can use the transformation method.

The inverse transformation is found as follows:

$$u = h(v) = \sqrt{v} \implies v = u^2 = h^{-1}(u).$$

Also, the derivative of the inverse transformation is

$$\frac{d}{du}h^{-1}(u) = \frac{d}{du}u^2 = 2u.$$

Therefore, for $u > 0$, the pdf of U is

$$f_U(u) = f_V(h^{-1}(u)) \left| \frac{d}{du}h^{-1}(u) \right| = \frac{1}{2}e^{-u^2/2} \times |2u| = ue^{-u^2/2}.$$

Summarizing, the pdf of $U = h(V) = \sqrt{V}$ is

$$f_U(u) = \begin{cases} ue^{-u^2/2}, & u > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Comparing this pdf to the general form of the Weibull(m, θ) pdf

$$f_U(u) = \begin{cases} \frac{m}{\theta}u^{m-1}e^{-u^m/\theta}, & u > 0 \\ 0, & \text{otherwise,} \end{cases}$$

we see that $U \sim$ Weibull$(m = 2, \theta = 2)$. This pdf is shown above.

6.52. (a) We did this part in Example 6.11 of the notes. Suppose $Y_1 \sim \text{Poisson}(\lambda_1)$ and $Y_2 \sim \text{Poisson}(\lambda_2)$. If Y_1 and Y_2 are independent, the mgf of $U = Y_1 + Y_2$ is

$$m_U(t) = m_{Y_1}(t)m_{Y_2}(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}.$$

We recognize this as the mgf of a Poisson random variable with mean $\lambda_1 + \lambda_2$. Because mgfs are unique, we know that $U \sim \text{Poisson}(\lambda_1 + \lambda_2)$. The pmf of U is

$$p_U(u) = \begin{cases} \dfrac{(\lambda_1+\lambda_2)^u e^{-(\lambda_1+\lambda_2)}}{u!}, & u = 0, 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

(b) In this part, we want to find $p_{Y_1|U}(y_1|m)$, the conditional pmf of Y_1, given $U = Y_1 + Y_2 = m$. First note that if the sum $U = Y_1 + Y_2 = m$, then the possible values of Y_1 are $\{y_1 : y_1 = 0, 1, 2, ..., m\}$. Therefore, the conditional pmf $p_{Y_1|U}(y_1|m)$ is nonzero for these values of y_1, and is otherwise equal to zero. Recall from STAT 511, the definition of a conditional pmf

$$p_{Y_1|U}(y_1|m) = \frac{p_{Y_1,U}(y_1, m)}{p_U(m)} = \frac{P(Y_1 = y_1, U = m)}{P(U = m)}.$$

We know

$$P(U = m) = p_U(m) = \frac{(\lambda_1+\lambda_2)^m e^{-(\lambda_1+\lambda_2)}}{m!}$$

from part (a). How do we find the joint probability $P(Y_1 = y_1, U = m)$? We don't have the joint pmf of Y_1 and U, so it is not clear how to calculate this. The key is to note that

$$\{Y_1 = y_1, U = m\} \ = \ \{Y_1 = y_1, Y_1 + Y_2 = m\} \ = \ \{Y_1 = y_1, Y_2 = m - y_1\}.$$

Therefore,

$$P(Y_1 = y_1, U = m) \ = \ P(Y_1 = y_1, Y_2 = m - y_1) \ \overset{Y_1 \perp\!\!\!\perp Y_2}{=} \ P(Y_1 = y_1)P(Y_2 = m - y_1).$$

We can calculate these two probabilities because $Y_1 \sim \text{Poisson}(\lambda_1)$ and $Y_2 \sim \text{Poisson}(\lambda_2)$; that is,

$$P(Y_1 = y_1) = \frac{\lambda_1^{y_1} e^{-\lambda_1}}{y_1!} \quad \text{and} \quad P(Y_2 = m - y_1) = \frac{\lambda_2^{m-y_1} e^{-\lambda_2}}{(m-y_1)!}.$$

Therefore,

$$\begin{aligned} p_{Y_1|U}(y_1|m) &= \frac{P(Y_1 = y_1)P(Y_2 = m - y_1)}{P(U = m)} \\ &= \frac{\dfrac{\lambda_1^{y_1} e^{-\lambda_1}}{y_1!} \dfrac{\lambda_2^{m-y_1} e^{-\lambda_2}}{(m-y_1)!}}{\dfrac{(\lambda_1+\lambda_2)^m e^{-(\lambda_1+\lambda_2)}}{m!}} \\ &= \frac{m!}{y_1!(m-y_1)!} \frac{\lambda_1^{y_1}}{(\lambda_1+\lambda_2)^{y_1}} \frac{\lambda_2^{m-y_1}}{(\lambda_1+\lambda_2)^{m-y_1}} \\ &= \binom{m}{y_1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y_1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{m-y_1} = \binom{m}{y_1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y_1}\left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{m-y_1}. \end{aligned}$$

Summarizing,

$$p_{Y_1|U}(y_1|m) = \begin{cases} \binom{m}{y_1}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y_1}\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{m-y_1}, & y_1 = 0,1,2,...,m \\ 0, & \text{otherwise.} \end{cases}$$

We recognize this as the pmf of a binomial random variable with number of trials m and success probability

$$p = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

Therefore, we have shown

$$Y_1 \sim \text{Poisson}(\lambda_1), Y_2 \sim \text{Poisson}(\lambda_2), Y_1 \perp\!\!\!\perp Y_2 \implies Y_1|Y_1+Y_2 = m \sim b\left(m, \frac{\lambda_1}{\lambda_1+\lambda_2}\right).$$

6.57. We are given

$$\begin{aligned} Y_1 &\sim \text{gamma}(\alpha_1, \beta) \\ Y_2 &\sim \text{gamma}(\alpha_2, \beta) \\ &\vdots \\ Y_n &\sim \text{gamma}(\alpha_n, \beta) \end{aligned}$$

and $Y_1, Y_2, ..., Y_n$ are mutually independent. We want to find the distribution of

$$U = Y_1 + Y_2 + \cdots + Y_n.$$

Whenever you are asked to find the distribution of the sum of mutually independent random variables, try the mgf method. The mgf of the sum U is

$$\begin{aligned} m_U(t) &= m_{Y_1}(t)m_{Y_2}(t)\cdots m_{Y_n}(t) \\ &= \left(\frac{1}{1-\beta t}\right)^{\alpha_1} \times \left(\frac{1}{1-\beta t}\right)^{\alpha_2} \times \cdots \times \left(\frac{1}{1-\beta t}\right)^{\alpha_n} = \left(\frac{1}{1-\beta t}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_n}. \end{aligned}$$

We recognize this as the mgf of a gamma random variable with shape parameter $\alpha_1+\alpha_2+\cdots+\alpha_n$ and scale parameter β. Because mgfs are unique, we know

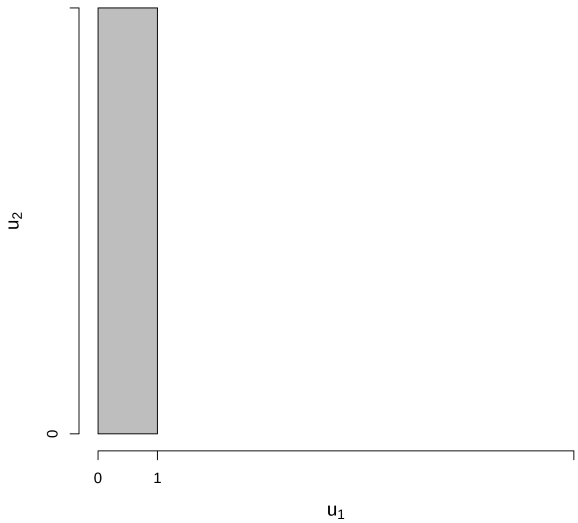
$$U = Y_1 + Y_2 + \cdots + Y_n \sim \text{gamma}(\alpha_1 + \alpha_2 + \cdots + \alpha_n, \beta).$$

6.59. We are given $Y_1 \sim \chi^2(\nu_1)$, $Y_2 \sim \chi^2(\nu_2)$, and Y_1 and Y_2 are independent. We want to find the distribution of $U = Y_1 + Y_2$. Use the mgf method. The mgf of the sum U is

$$m_U(t) = m_{Y_1}(t)m_{Y_2}(t) = \left(\frac{1}{1-2t}\right)^{\nu_1/2}\left(\frac{1}{1-2t}\right)^{\nu_2/2} = \left(\frac{1}{1-2t}\right)^{(\nu_1+\nu_2)/2}.$$

We recognize this as the mgf of a χ^2 random variable with degrees of freedom $\nu_1 + \nu_2$. Because mgfs are unique, we know $U = Y_1 + Y_2 \sim \chi^2(\nu_1 + \nu_2)$.

Note: See how easy the mgf method is? As an exercise, try to redo Exercise 6.59 by using the cdf method; i.e., derive $F_U(u) = P(U \leq u)$ directly and then take derivatives. You should get the $\chi^2(\nu_1 + \nu_2)$ pdf. This argument is much harder, but it still should work.



6.63. The authors have already done the bivariate transformation for us. Starting with $Y_1 \sim$ exponential(β), $Y_2 \sim$ exponential(β), and $Y_1 \perp\!\!\!\perp Y_2$, the authors show the joint distribution of

$$U_1 = \frac{Y_1}{Y_1 + Y_2} \quad \text{and} \quad U_2 = Y_1 + Y_2$$

is

$$f_{U_1,U_2}(u_1,u_2) = \begin{cases} \dfrac{1}{\beta^2} u_2 e^{-u_2/\beta}, & 0 < u_1 < 1,\ u_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Go through the bivariate transformation again and re-derive this yourself for practice. Note the support of (U_1, U_2) is

$$R_{U_1,U_2} = \{(u_1,u_2) : 0 < u_1 < 1,\ u_2 > 0\}.$$

This region is shown above. The joint pdf $f_{U_1,U_2}(u_1,u_2)$ is a three-dimensional function which takes the value $\frac{1}{\beta^2} u_2 e^{-u_2/\beta}$ over this region and is otherwise equal to zero.

(a) To find the marginal distribution of U_1, we integrate the joint pdf $f_{U_1,U_2}(u_1,u_2)$ over u_2. For $0 < u_1 < 1$, we have

$$f_{U_1}(u_1) = \int_{u_2=0}^{\infty} f_{U_1,U_2}(u_1,u_2) du_2 = \int_{u_2=0}^{\infty} \frac{1}{\beta^2} u_2 e^{-u_2/\beta} du_2 = 1,$$

because $\frac{1}{\beta^2} u_2 e^{-u_2/\beta}$ is the gamma$(2,\beta)$ pdf and we are integrating over $(0,\infty)$. We have shown

$$f_{U_1}(u_1) = \begin{cases} 1, & 0 < u_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

We recognize this as the $\mathcal{U}(0,1)$ pdf; i.e., $U_1 \sim \mathcal{U}(0,1)$.

(b) To find the marginal distribution of U_2, we integrate the joint pdf $f_{U_1,U_2}(u_1,u_2)$ over u_1. For $u_2 > 0$, we have

$$\begin{aligned} f_{U_2}(u_2) = \int_{u_1=0}^{1} f_{U_1,U_2}(u_1,u_2)du_1 &= \int_{u_1=0}^{1} \frac{1}{\beta^2}u_2 e^{-u_2/\beta}du_1 \\ &= \frac{1}{\beta^2}u_2 e^{-u_2/\beta}\int_{u_1=0}^{1} 1du_1 = \frac{1}{\beta^2}u_2 e^{-u_2/\beta}. \end{aligned}$$

We have shown

$$f_{U_2}(u_2) = \begin{cases} \dfrac{1}{\beta^2}u_2 e^{-u_2/\beta}, & u_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

We recognize this as the gamma$(2,\beta)$ pdf; i.e., $U_2 \sim \text{gamma}(2,\beta)$.

(c) Note that we can write

$$f_{U_1,U_2}(u_1,u_2) = \frac{1}{\beta^2}u_2 e^{-u_2/\beta} = 1 \times \frac{1}{\beta^2}u_2 e^{-u_2/\beta} = f_{U_1}(u_1)f_{U_2}(u_2).$$

Because the joint pdf can be written as the product of the marginal pdfs, we know $U_1 \perp\!\!\!\perp U_2$.

6.68. We start with the random variables Y_1 and Y_2, whose joint pdf is

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 8y_1y_2, & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note the support of (Y_1,Y_2) is

$$R_{Y_1,Y_2} = \{(y_1,y_2) : 0 \le y_1 \le y_2 \le 1\}.$$

The graph of R_{Y_1,Y_2} is shown at the top of the next page (left). The joint pdf $f_{Y_1,Y_2}(y_1,y_2)$ is a three-dimensional function which takes the value $8y_1y_2$ over this triangular region and is otherwise equal to zero.

Our goal is to find the joint pdf of

$$\begin{aligned} U_1 = h_1(Y_1,Y_2) &= \frac{Y_1}{Y_2} \\ U_2 = h_2(Y_1,Y_2) &= Y_2. \end{aligned}$$

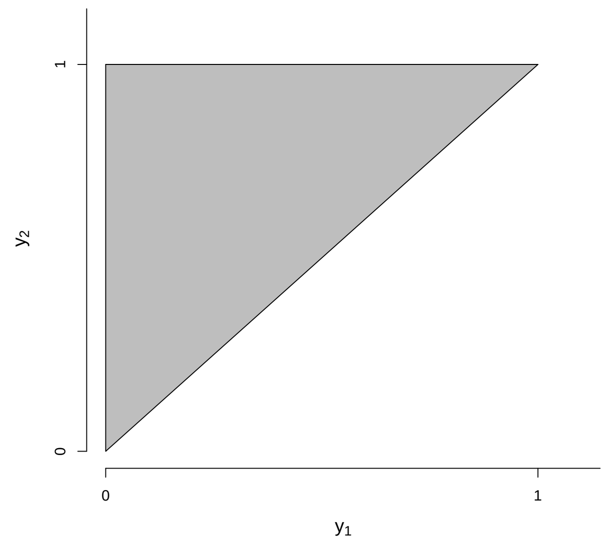
We use a bivariate transformation. We first find the support of (U_1,U_2). Note that

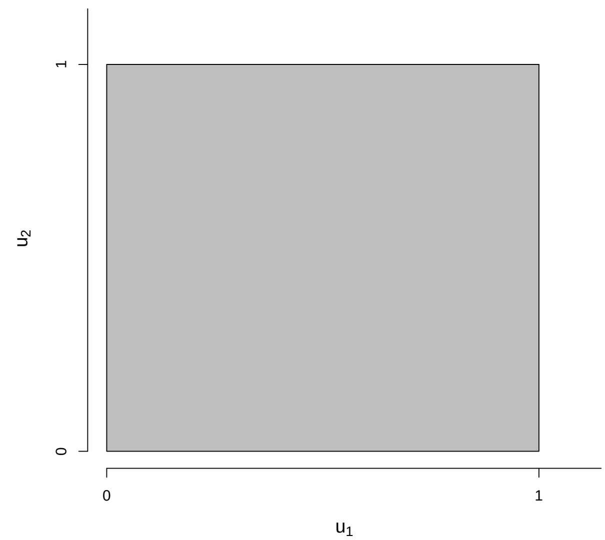
$$0 \le y_1 \le y_2 \le 1 \implies u_1 = \frac{y_1}{y_2} \in [0,1]$$

and $0 \le u_2 = y_2 \le 1$. Therefore, the support of (U_1,U_2) is

$$R_{U_1,U_2} = \{(u_1,u_2) : 0 \le u_1 \le 1,\ 0 \le u_2 \le 1\}.$$

The graph of R_{U_1,U_2} is shown at the top of the next page (right).





To verify the transformation above is one-to-one, we show $h(y_1, y_2) = h(y_1^*, y_2^*) \Longrightarrow y_1 = y_1^*$ and $y_2 = y_2^*$, where

$$h\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} \dfrac{y_1}{y_2} \\ y_2 \end{pmatrix}.$$

Suppose $h(y_1, y_2) = h(y_1^*, y_2^*)$. Clearly $y_2 = y_2^*$. Then the first equation implies $y_1 = y_1^*$. Therefore the transformation is one to one.

The inverse transformation is found by solving

$$\begin{aligned} u_1 &= \frac{y_1}{y_2} \\ u_2 &= y_2 \end{aligned}$$

for $y_1 = h_1^{-1}(u_1, u_2)$ and $y_2 = h_2^{-1}(u_1, u_2)$. Straightforward algebra shows

$$\begin{aligned} y_1 &= h_1^{-1}(u_1, u_2) = u_1 u_2 \\ y_2 &= h_2^{-1}(u_1, u_2) = u_2. \end{aligned}$$

The Jacobian is

$$J = \det \begin{vmatrix} \dfrac{\partial h_1^{-1}(u_1, u_2)}{\partial u_1} & \dfrac{\partial h_1^{-1}(u_1, u_2)}{\partial u_2} \\ \dfrac{\partial h_2^{-1}(u_1, u_2)}{\partial u_1} & \dfrac{\partial h_2^{-1}(u_1, u_2)}{\partial u_2} \end{vmatrix} = \det \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2(1) - u_1(0) = u_2.$$

Therefore, the joint pdf of (U_1, U_2), where nonzero, is

$$\begin{aligned} f_{U_1,U_2}(u_1, u_2) &= f_{Y_1,Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))|J| \\ &= f_{Y_1,Y_2}(u_1 u_2, u_2)|u_2| \\ &= 8(u_1 u_2) u_2 \times u_2 \\ &= 8 u_1 u_2^3. \end{aligned}$$

Summarizing, the joint pdf of (U_1, U_2) is

$$f_{U_1,U_2}(u_1,u_2) = \begin{cases} 8u_1u_2^3, & 0 \le u_1 \le 1, 0 \le u_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Note that we can write

$$f_{U_1,U_2}(u_1,u_2) = 8u_1u_2^3 = 2u_1 \times 4u_2^3 = f_{U_1}(u_1)f_{U_2}(u_2).$$

We recognize

$$f_{U_1}(u_1) = \begin{cases} 2u_1, & 0 \le u_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_{U_2}(u_2) = \begin{cases} 4u_2^3, & 0 \le u_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

as beta pdfs. Specifically, $U_1 \sim \text{beta}(2,1)$ and $U_2 \sim \text{beta}(4,1)$. Because the joint pdf can be written as the product of the marginal pdfs, we know $U_1 \perp\!\!\!\perp U_2$.