6.66. We start with Y_1 and Y_2 which have the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$. Define

$$\begin{aligned} U_1 &= h_1(Y_1, Y_2) = Y_1 + Y_2 \\ U_2 &= h_2(Y_1, Y_2) = Y_2. \end{aligned}$$

(a) Note that

$$\begin{aligned} u_1 &= h_1(y_1, y_2) = y_1 + y_2 \\ u_2 &= h_2(y_1, y_2) = y_2 \end{aligned}$$

is a linear transformation; i.e.,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{A}\mathbf{y},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The transformation is 1:1 because $\mathbf{A}^{-1}$ exists. Because the transformation is 1:1, the inverse transformation exists and is given by

$$\begin{aligned} y_1 &= h_1^{-1}(y_1, y_2) = u_1 - u_2 \\ y_2 &= h_2^{-1}(u_1, u_2) = u_2. \end{aligned}$$

The Jacobian is

$$J = \det \begin{vmatrix} \dfrac{\partial h_1^{-1}(u_1, u_2)}{\partial u_1} & \dfrac{\partial h_1^{-1}(u_1, u_2)}{\partial u_2} \\ \dfrac{\partial h_2^{-1}(u_1, u_2)}{\partial u_1} & \dfrac{\partial h_2^{-1}(u_1, u_2)}{\partial u_2} \end{vmatrix} = \det \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1(1) - (-1)(0) = 1.$$

Therefore, the joint pdf of $\mathbf{U} = (U_1, U_2)$, where nonzero, is

$$\begin{aligned} f_{U_1,U_2}(u_1, u_2) &= f_{Y_1,Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))|J| \\ &= f_{Y_1,Y_2}(u_1 - u_2, u_2)|1| \\ &= f_{Y_1,Y_2}(u_1 - u_2, u_2). \end{aligned}$$

(b) The marginal pdf of U_1 is obtained by taking the joint pdf $f_{U_1,U_2}(u_1, u_2)$ and integrating over u_2. Therefore,

$$f_{U_1}(u_1) = \int_{\mathbb{R}} f_{U_1,U_2}(u_1, u_2) du_2 = \int_{\mathbb{R}} f_{Y_1,Y_2}(u_1 - u_2, u_2) du_2$$

as claimed.

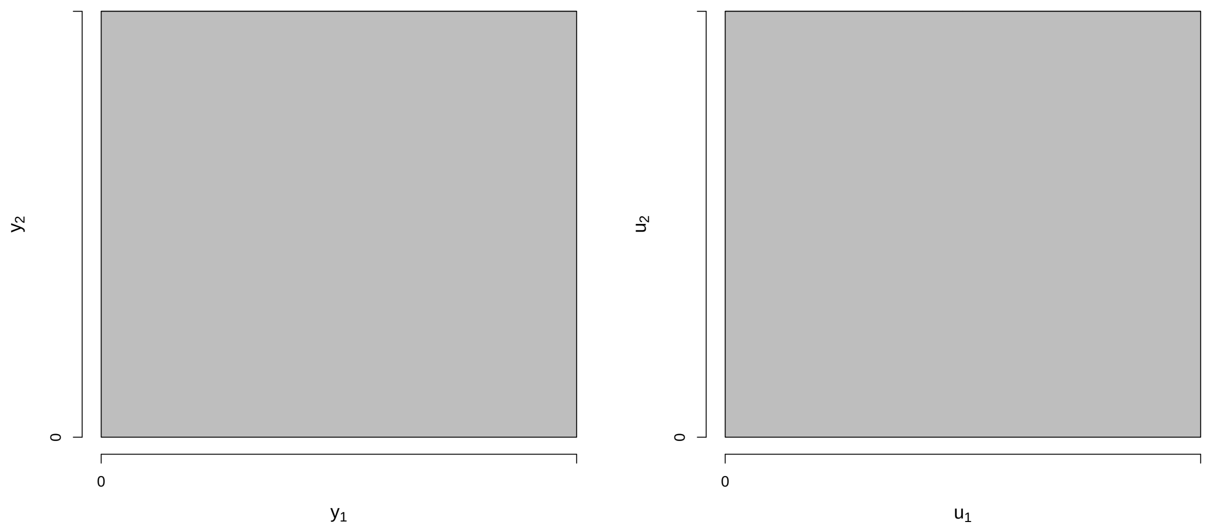
(c) If we additionally assume $Y_1 \perp\!\!\!\perp Y_2$, then we know

$$f_{Y_1,Y_2}(u_1 - u_2, u_2) = f_{Y_1}(u_1 - u_2) f_{Y_2}(u_2);$$

i.e., the joint pdf factors into the product of the marginal pdfs. Therefore, part (b) becomes

$$f_{U_1}(u_1) = \int_{\mathbb{R}} f_{Y_1}(u_1 - u_2) f_{Y_2}(u_2) du_2$$

as claimed.



Remark: The formula

$$f_{U_1}(u_1) = \int_{\mathbb{R}} f_{Y_1}(u_1 - u_2) f_{Y_2}(u_2) du_2$$

is called the **convolution formula** to derive the pdf of $U_1 = Y_1 + Y_2$. This formula, in theory, can always be used to derive the pdf of the sum of independent continuous random variables (there is a discrete version as well that convolves pmfs). Of course, if the goal is to derive the distribution of the sum $Y_1 + Y_2$, then the mgf method is so much easier. However, the mgf method does not always work; e.g., mgfs may not exist, the mgf of the sum may not be one that we recognize, etc. The cdf method is always available too to derive the cdf of $U_1 = Y_1 + Y_2$ and then take derivatives to get the pdf. I think of using the convolution formula as a "last resort," but it does work. I have also seen Exam P problems that ask students to apply the convolution method.

6.71. We start with independent random variables $Y_1 \sim$ exponential(β) and $Y_2 \sim$ exponential(β). For $y_1 > 0$ and $y_2 > 0$, the joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) = \frac{1}{\beta} e^{-y_1/\beta} \times \frac{1}{\beta} e^{-y_2/\beta} = \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta}.$$

Summarizing,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta}, & y_1 > 0, y_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note the support of (Y_1, Y_2) is

$$R_{Y_1,Y_2} = \{(y_1, y_2) : y_1 > 0, y_2 > 0\};$$

i.e., the entire first quadrant. The graph of R_{Y_1,Y_2} is shown at the top of the last page (left). The joint pdf $f_{Y_1,Y_2}(y_1,y_2)$ is a three-dimensional function which takes the value $\frac{1}{\beta^2}e^{-(y_1+y_2)/\beta}$ over this region and is otherwise equal to zero. Our goal is to find the joint pdf of

$$\begin{aligned} U_1 = h_1(Y_1,Y_2) &= Y_1 + Y_2 \\ U_2 = h_2(Y_1,Y_2) &= \frac{Y_1}{Y_2}. \end{aligned}$$

We use a bivariate transformation. We first find the support of (U_1,U_2). Note that

$$y_1 > 0, y_2 > 0 \implies u_1 = y_1 + y_2 > 0$$

and $u_2 = y_1/y_2 > 0$ as well. Therefore, the support of (U_1,U_2) is

$$R_{U_1,U_2} = \{(u_1,u_2) : u_1 > 0, u_2 > 0\};$$

i.e., also the entire first quadrant; see the top of the last page (right). That is, the support of (Y_1,Y_2) and the support of (U_1,U_2) is the same set.

To verify the transformation above is one-to-one, we show $h(y_1,y_2) = h(y_1^*,y_2^*) \implies y_1 = y_1^*$ and $y_2 = y_2^*$, where

$$h\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1(y_1,y_2) \\ h_2(y_1,y_2) \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ \dfrac{y_1}{y_2} \end{pmatrix}.$$

Suppose $h(y_1,y_2) = h(y_1^*,y_2^*)$. The first equation implies $y_1+y_2 = y_1^*+y_2^* \iff y_1 = y_1^*+y_2^*-y_2$. Plugging this into the second equation implies

$$\begin{aligned} \frac{y_1^*+y_2^*-y_2}{y_2} = \frac{y_1^*}{y_2^*} &\iff y_1^*y_2^* + y_2^*y_2^* - y_2y_2^* = y_1^*y_2 \\ &\iff y_1^*y_2^* + y_2^*y_2^* = y_1^*y_2 + y_2y_2^* \\ &\iff y_2^*(y_1^*+y_2^*) = y_2(y_1^*+y_2^*) \iff y_2 = y_2^*. \end{aligned}$$

The first equation then implies $y_1 = y_1^*$. Therefore, the transformation is 1:1.

The inverse transformation is found by solving

$$\begin{aligned} u_1 &= y_1 + y_2 \\ u_2 &= \frac{y_1}{y_2}. \end{aligned}$$

for $y_1 = h_1^{-1}(u_1,u_2)$ and $y_2 = h_2^{-1}(u_1,u_2)$. The second equation implies $u_2y_2 = y_1$, so the first equation becomes

$$u_1 = u_2y_2 + y_2 \implies u_1 = y_2(1+u_2) \implies y_2 = \frac{u_1}{1+u_2}.$$

Therefore,

$$u_2 = \frac{y_1}{y_2} = \frac{y_1}{\dfrac{u_1}{1+u_2}} \implies y_1 = \frac{u_1u_2}{1+u_2}.$$

Summarizing, we have

$$\begin{aligned} y_1 = h_1^{-1}(u_1,u_2) &= \frac{u_1u_2}{1+u_2} \\ y_2 = h_2^{-1}(u_1,u_2) &= \frac{u_1}{1+u_2}. \end{aligned}$$

The Jacobian is

$$
\begin{aligned}
J = \det \begin{vmatrix} \dfrac{\partial h_1^{-1}(u_1,u_2)}{\partial u_1} & \dfrac{\partial h_1^{-1}(u_1,u_2)}{\partial u_2} \\ \dfrac{\partial h_2^{-1}(u_1,u_2)}{\partial u_1} & \dfrac{\partial h_2^{-1}(u_1,u_2)}{\partial u_2} \end{vmatrix} &= \det \begin{vmatrix} \dfrac{u_2}{1+u_2} & \dfrac{u_1}{(1+u_2)^2} \\ \dfrac{1}{1+u_2} & -\dfrac{u_1}{(1+u_2)^2} \end{vmatrix} \\
&= \frac{u_2}{1+u_2}\left[-\frac{u_1}{(1+u_2)^2}\right] - \frac{u_1}{(1+u_2)^2}\left(\frac{1}{1+u_2}\right) \\
&= -\frac{u_1u_2}{(1+u_2)^3} - \frac{u_1}{(1+u_2)^3} \\
&= -\frac{u_1(1+u_2)}{(1+u_2)^3} = -\frac{u_1}{(1+u_2)^2}
\end{aligned}
$$

Therefore, the joint pdf of $\mathbf{U} = (U_1, U_2)$, where nonzero, is

$$
\begin{aligned}
f_{U_1,U_2}(u_1,u_2) &= f_{Y_1,Y_2}(h_1^{-1}(u_1,u_2), h_2^{-1}(u_1,u_2))|J| \\
&= f_{Y_1,Y_2}\left(\frac{u_1u_2}{1+u_2}, \frac{u_1}{1+u_2}\right)\left|-\frac{u_1}{(1+u_2)^2}\right|.
\end{aligned}
$$

Note that

$$
f_{Y_1,Y_2}\left(\frac{u_1u_2}{1+u_2}, \frac{u_1}{1+u_2}\right) = \frac{1}{\beta^2}e^{-\left(\frac{u_1u_2}{1+u_2}+\frac{u_1}{1+u_2}\right)/\beta} = \frac{1}{\beta^2}e^{-\left(\frac{u_1u_2+u_1}{1+u_2}\right)/\beta} = \frac{1}{\beta^2}e^{-u_1\left(\frac{1+u_2}{1+u_2}\right)/\beta} = \frac{1}{\beta^2}e^{-u_1/\beta}.
$$

Therefore, for $u_1 > 0$ and $u_2 > 0$,

$$
f_{U_1,U_2}(u_1,u_2) = \frac{1}{\beta^2}e^{-u_1/\beta}\left|-\frac{u_1}{(1+u_2)^2}\right| = \frac{1}{\beta^2}u_1e^{-u_1/\beta}\frac{1}{(1+u_2)^2}.
$$

Summarizing,

$$
f_{U_1,U_2}(u_1,u_2) = \begin{cases} \dfrac{1}{\beta^2}u_1e^{-u_1/\beta}\dfrac{1}{(1+u_2)^2}, & u_1 > 0, u_2 > 0 \\ 0, & \text{otherwise.} \end{cases}
$$

(b) Note that we can write

$$
f_{U_1,U_2}(u_1,u_2) = \underbrace{\frac{1}{\beta^2}u_1e^{-u_1/\beta}}_{g_1(u_1)}\underbrace{\frac{1}{(1+u_2)^2}}_{g_2(u_2)};
$$

that is, we can factor the joint pdf $f_{U_1,U_2}(u_1,u_2)$ into the product of two nonnegative functions $g_1(u_1)$ and $g_2(u_2)$. By Theorem 5.5 (WMS, pp 250), U_1 and U_2 are independent. You should be able to see that $U_1 \sim \text{gamma}(2,\beta)$. I think U_2 has a Pareto-type distribution with a horizontal shift.

6.74. We are given $Y_1, Y_2, ..., Y_n$ are mutually independent and identically distributed (i.e., iid) $\mathcal{U}(0,\theta)$ random variables, where $\theta > 0$. Recall the $\mathcal{U}(0,\theta)$ pdf is given by

$$
f_Y(y) = \begin{cases} \dfrac{1}{\theta}, & 0 < y < \theta \\ 0, & \text{otherwise} \end{cases}
$$

and the $\mathcal{U}(0,\theta)$ cdf is

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ \dfrac{y}{\theta}, & 0 < y < \theta \\ 1, & y \ge \theta. \end{cases}$$

(a) They are asking for the cdf of $Y_{(n)}$, the maximum order statistic. We derived this in the notes; in general,

$$\begin{aligned} F_{Y_{(n)}}(y) = P(Y_{(n)} \le y) &= P(Y_1 \le y, Y_2 \le y, ..., Y_n \le y) \\ &= P(Y_1 \le y)P(Y_2 \le y)\cdots P(Y_n \le y) \\ &= [P(Y \le y)]^n \\ &= [F_Y(y)]^n. \end{aligned}$$

Therefore,

$$F_{Y_{(n)}}(y) = \begin{cases} 0, & y \le 0 \\ \left(\dfrac{y}{\theta}\right)^n, & 0 < y < \theta \\ 1, & y \ge \theta. \end{cases}$$

(b) They are asking for the pdf of $Y_{(n)}$. For $0 < y < \theta$, we have

$$f_{Y_{(n)}}(y) = nf_Y(y)[F_Y(y)]^{n-1} = n\left(\frac{1}{\theta}\right)\left(\frac{y}{\theta}\right)^{n-1} = \frac{ny^{n-1}}{\theta^n}.$$

Summarizing,

$$f_{Y_{(n)}}(y) = \begin{cases} \dfrac{ny^{n-1}}{\theta^n}, & 0 < y < \theta \\ 0, & \text{otherwise.} \end{cases}$$

(c) The mean of $Y_{(n)}$ is

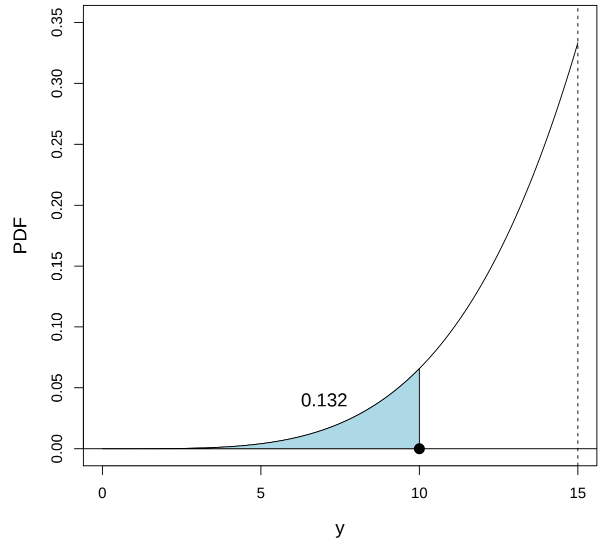
$$\begin{aligned} E(Y_{(n)}) = \int_{\mathbb{R}} y f_{Y_{(n)}}(y)dy &= \int_0^\theta y\ \frac{ny^{n-1}}{\theta^n}dy \\ &= \frac{n}{\theta^n}\int_0^\theta y^n dy = \frac{n}{\theta^n}\left(\frac{y^{n+1}}{n+1}\right)\Bigg|_0^\theta = \frac{n}{n+1}\frac{\theta^{n+1}}{\theta^n} = \left(\frac{n}{n+1}\right)\theta. \end{aligned}$$

The second moment of $Y_{(n)}$ is

$$\begin{aligned} E(Y_{(n)}^2) = \int_{\mathbb{R}} y^2 f_{Y_{(n)}}(y)dy &= \int_0^\theta y^2\ \frac{ny^{n-1}}{\theta^n}dy \\ &= \frac{n}{\theta^n}\int_0^\theta y^{n+1} dy = \frac{n}{\theta^n}\left(\frac{y^{n+2}}{n+2}\right)\Bigg|_0^\theta = \frac{n}{n+2}\frac{\theta^{n+2}}{\theta^n} = \left(\frac{n}{n+2}\right)\theta^2. \end{aligned}$$

Therefore, the variance of $Y_{(n)}$ is

$$\begin{aligned} V(Y_{(n)}) = E(Y_{(n)}^2) - [E(Y_{(n)})]^2 &= \left(\frac{n}{n+2}\right)\theta^2 - \left[\left(\frac{n}{n+1}\right)\theta\right]^2 \\ &= \left[\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right]\theta^2 = \left[\frac{n}{(n+1)^2(n+2)}\right]\theta^2. \end{aligned}$$



6.75. Let Y_1, Y_2, Y_3, Y_4, Y_5 denote your waiting times for the bus (in minutes). Individual waiting times are distributed as $\mathcal{U}(0, 15)$. Assume Y_1, Y_2, Y_3, Y_4, Y_5 are mutually independent. From Exercise 6.74, the pdf of $Y_{(5)}$, the longest waiting time (i.e., the maximum of the 5 times) is

$$f_{Y_{(5)}}(y) = \begin{cases} \dfrac{5y^4}{15^5}, & 0 < y < 15 \\ 0, & \text{otherwise.} \end{cases}$$

This pdf is shown above. The probability your longest wait time $Y_{(5)}$ is less than 10 minutes is

$$P(Y_{(5)} < 10) = \int_0^{10} f_{Y_{(5)}}(y)dy = \int_0^{10} \frac{5y^4}{15^5}dy = \frac{1}{15^5}\left(y^5\Big|_0^{10}\right) = \frac{10^5}{15^5} \approx 0.132.$$

The probability $P(Y_{(5)} < 10)$ is shaded above.

6.84. Suppose $Y_1, Y_2, ..., Y_n$ are mutually independent random variables, all with the same Weibull(m, α) distribution; i.e., $Y_1, Y_2, ..., Y_n$ are iid Weibull(m, α). **Note:** The phrase "random sample" means "iid." The Weibull(m, α) pdf is

$$f_Y(y) = \begin{cases} \dfrac{m}{\alpha}y^{m-1}e^{-y^m/\alpha}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The Weibull(m, α) cdf is

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y^m/\alpha}, & y > 0. \end{cases}$$

To see where the cdf comes from, let's derive it. Clearly, $F_Y(y) = 0$, when $y \leq 0$. For $y > 0$, we have

$$F_Y(y) = P(Y \leq y) = \int_0^y \frac{m}{\alpha}t^{m-1}e^{-t^m/\alpha}dt.$$

In the last integral, let

$$u = t^m \implies du = mt^{m-1}dt.$$

The limits change under this transformation; as $t : 0 \to y$, we have $u : 0 \to y^m$. Therefore, the last integral becomes

$$\int_0^y \frac{m}{\alpha} t^{m-1} e^{-t^m/\alpha} dt = \int_0^{y^m} \frac{m}{\alpha} t^{m-1} e^{-u/\alpha} \frac{du}{mt^{m-1}} = \int_0^{y^m} \frac{1}{\alpha} e^{-u/\alpha} du,$$

which is the exponential(α) cdf evaluated at $u = y^m$. Therefore, the result.

We find to find the pdf of $Y_{(1)}$, the minimum order statistic. Recall in general,

$$f_{Y_{(1)}}(y) = nf_Y(y)[1 - F_Y(y)]^{n-1}.$$

Therefore, for $y > 0$, we have

$$\begin{aligned} f_{Y_{(1)}}(y) &= n\left(\frac{m}{\alpha} y^{m-1} e^{-y^m/\alpha}\right)\left[1 - (1 - e^{-y^m/\alpha})\right]^{n-1} \\ &= n\left(\frac{m}{\alpha} y^{m-1} e^{-y^m/\alpha}\right)\left(e^{-y^m/\alpha}\right)^{n-1} = \frac{mn}{\alpha} y^{m-1}\left(e^{-y^m/\alpha}\right)^n = \frac{mn}{\alpha} y^{m-1} e^{-ny^m/\alpha}. \end{aligned}$$

Summarizing,

$$f_{Y_{(1)}}(y) = \begin{cases} \dfrac{mn}{\alpha} y^{m-1} e^{-ny^m/\alpha}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note: We can write the nonzero part of $f_{Y_{(1)}}(y)$ as

$$\frac{mn}{\alpha} y^{m-1} e^{-ny^m/\alpha} = \frac{m}{(\alpha/n)} y^{m-1} e^{-y^m/(\alpha/n)},$$

which we recognize as a Weibull pdf with parameters m and α/n. Therefore,

$$Y_1, Y_2, ..., Y_n \sim \text{iid Weibull}(m, \alpha) \implies Y_{(1)} \sim \text{Weibull}(m, \alpha/n).$$

6.87. We are given Y_1 and Y_2 (two stock opening prices), which have the common pdf

$$f_Y(y) = \begin{cases} \dfrac{1}{2} e^{-(y-4)/2}, & y \geq 4 \\ 0, & \text{otherwise.} \end{cases}$$

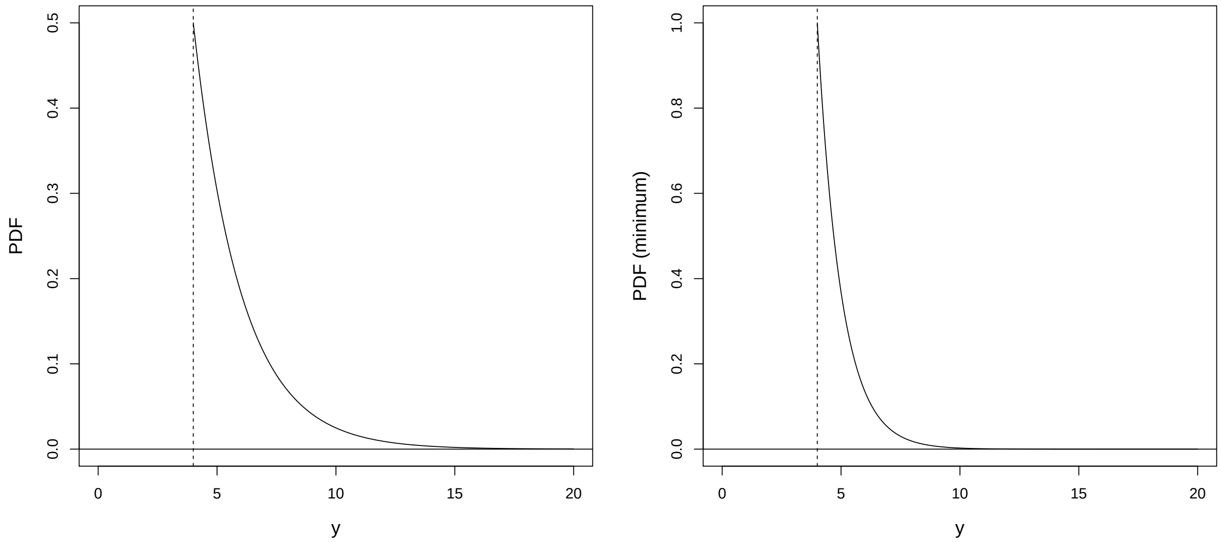
Note that this is an exponential(1/2) pdf with a horizontal shift of 4 units to the right; see the figure at the top of the next page (left).

(a) The common cdf is

$$F_Y(y) = \begin{cases} 0, & y < 4 \\ 1 - e^{-(y-4)/2}, & y \geq 4. \end{cases}$$

To see where the cdf comes from, let's derive it. Clearly, $F_Y(y) = 0$, when $y < 4$. For $y \geq 4$, we have

$$F_Y(y) = P(Y \leq y) = \int_4^y \frac{1}{2} e^{-(t-4)/2} dt.$$



In the last integral, let

$$u = t - 4 \implies du = dt.$$

The limits change under this transformation; as $t : 4 \to y$, we have $u : 0 \to y - 4$. Therefore, the last integral becomes

$$\int_4^y \frac{1}{2} e^{-(t-4)/2} dt = \int_0^{y-4} \frac{1}{2} e^{-u/2} du = \frac{1}{2} \left(-2e^{-u/2} \Big|_0^{y-4} \right) = \left(e^{-u/2} \Big|_{y-4}^{0} \right) = 1 - e^{-(y-4)/2}.$$

The investor is going to buy the stock that is less expensive at the opening; therefore, s/he is going pay $Y_{(1)} = \min\{Y_1, Y_2\}$. Therefore, we want to find the pdf of the minimum order statistic $Y_{(1)}$, where Y_1, Y_2 are iid from $f_Y(y)$. For $y \geq 4$, we have

$$\begin{aligned} f_{Y_{(1)}}(y) = n f_Y(y)[1 - F_Y(y)]^{n-1} &= 2 \left[\frac{1}{2} e^{-(y-4)/2} \right] \left[1 - (1 - e^{-(y-4)/2}) \right]^{2-1} \\ &= e^{-(y-4)/2} e^{-(y-4)/2} = e^{-(y-4)}. \end{aligned}$$

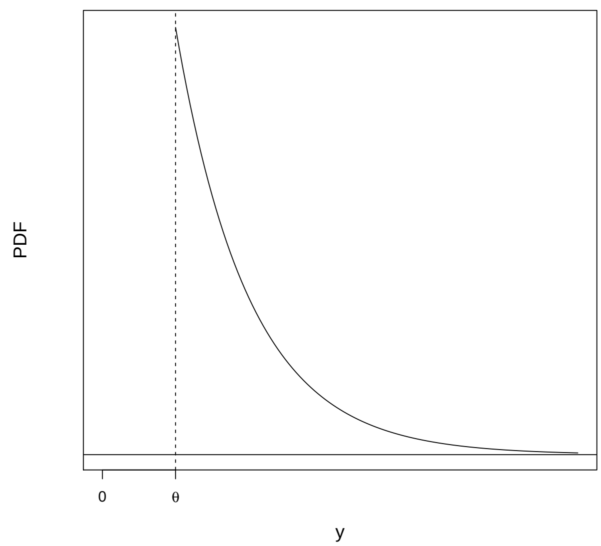
Summarizing,

$$f_{Y_{(1)}}(y) = \begin{cases} e^{-(y-4)}, & y \geq 4 \\ 0, & \text{otherwise.} \end{cases}$$

Note that this is an exponential(1) pdf with a horizontal shift of 4 units to the right; see the figure at the top of this page (right).

(b) We want to find the expected cost per share s/he will pay; i.e., we want to find $E(Y_{(1)})$. The mean of $Y_{(1)}$ is

$$E(Y_{(1)}) = \int_{\mathbb{R}} y f_{Y_{(1)}}(y) dy = \int_4^{\infty} y e^{-(y-4)} dy.$$



In the last integral, let

$$u = y - 4 \implies du = dy.$$

The limits change under this transformation; as $y : 4 \to \infty$, we have $u : 0 \to \infty$. Therefore, the last integral becomes

$$\int_4^\infty ye^{-(y-4)}dy = \int_0^\infty (u+4)e^{-u}du = E(U+4),$$

where $U \sim$ exponential(1). Therefore, $E(Y_{(1)}) = E(U+4) = E(U) + 4 = 1 + 4 = 5$.

6.88. We are given $Y_1, Y_2, ..., Y_n$ are iid from a shifted exponential distribution; i.e., the common pdf is

$$f_Y(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{otherwise.} \end{cases}$$

Note that this is the exponential(1) pdf shifted θ units to the right. In this example, $\theta > 0$ denotes the minimum time the worker needs to complete a task. More generally, θ is called the location or shift parameter. The pdf of Y is shown above.

(a) They are asking for the cdf of $Y_{(1)}$, the minimum order statistic. Therefore, we need to first derive the cdf of Y. Clearly $F_Y(y) = 0$ when $y \le \theta$. For $y > \theta$, we have

$$F_Y(y) = P(Y \le y) = \int_\theta^y e^{-(t-\theta)}dt.$$

In the last integral, let

$$u = t - \theta \implies du = dt.$$

The limits change under this transformation; as $t : \theta \to y$, we have $u : 0 \to y - \theta$. Therefore, the last integral becomes

$$\int_\theta^y e^{-(t-\theta)}dt = \int_0^{y-\theta} e^{-u}du = \left(-e^{-u}\big|_0^{y-\theta}\right) = 1 - e^{-(y-\theta)}.$$

Summarizing,

$$F_Y(y) = \begin{cases} 0, & y \le \theta \\ 1 - e^{-(y-\theta)}, & y > \theta. \end{cases}$$

For $y > \theta$, the pdf of $Y_{(1)}$ is

$$\begin{aligned} f_{Y_{(1)}}(y) = nf_Y(y)[1 - F_Y(y)]^{n-1} &= ne^{-(y-\theta)}\left\{1 - [1 - e^{-(y-\theta)}]\right\}^{n-1} \\ &= ne^{-(y-\theta)}\left[e^{-(y-\theta)}\right]^{n-1} \\ &= n\left[e^{-(y-\theta)}\right]^n = ne^{-n(y-\theta)}. \end{aligned}$$

Summarizing,

$$f_{Y_{(1)}}(y) = \begin{cases} ne^{-n(y-\theta)}, & y > \theta \\ 0, & \text{otherwise.} \end{cases}$$

Note that this is an exponential$(1/n)$ pdf with a horizontal shift of θ units to the right.

(b) The mean of $Y_{(1)}$ is

$$E(Y_{(1)}) = \int_{\mathbb{R}} y f_{Y_{(1)}}(y)dy = \int_\theta^\infty y\ ne^{-n(y-\theta)}dy.$$

In the last integral, let

$$u = y - \theta \implies du = dy.$$

The limits change under this transformation; as $y : \theta \to \infty$, we have $u : 0 \to \infty$. Therefore, the last integral becomes

$$\int_\theta^\infty y\ ne^{-n(y-\theta)}dy = \int_0^\infty (u+\theta)ne^{-nu}du = E(U+\theta),$$

where $U \sim$ exponential$(1/n)$; note that ne^{-nu} is the exponential$(1/n)$ pdf and we are integrating over $(0,\infty)$. Therefore, $E(Y_{(1)}) = E(U+\theta) = E(U) + \theta = \frac{1}{n} + \theta$.

6.103. Suppose $Y_1 \sim \mathcal{N}(0,1)$, $Y_2 \sim \mathcal{N}(0,1)$, and $Y_1 \perp\!\!\!\perp Y_2$. The joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2) = \frac{1}{\sqrt{2\pi}}e^{-y_1^2/2} \times \frac{1}{\sqrt{2\pi}}e^{-y_2^2/2} = \frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2},$$

for all $(y_1,y_2) \in \mathbb{R}^2$. Summarizing,

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2}, & -\infty < y_1 < \infty,\ -\infty < y_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

To find the distribution of Y_1/Y_2, we will perform a bivariate transformation with

$$\begin{aligned} U_1 &= h_1(Y_1, Y_2) = \frac{Y_1}{Y_2} \\ U_2 &= h_2(Y_1, Y_2) = Y_2 \end{aligned}$$

to get $f_{U_1,U_2}(u_1, u_2)$, the joint pdf of U_1 and U_2. We will then integrate $f_{U_1,U_2}(u_1, u_2)$ over u_2 to get the (marginal) pdf of $U_1 = Y_1/Y_2$.

We first find the support of (U_1, U_2). Note that

$$-\infty < y_1 < \infty, \ -\infty < y_2 < \infty \quad \Longrightarrow \quad u_1 = \frac{y_1}{y_2} \in (-\infty, \infty)$$

and $u_2 = y_2 \in (-\infty, \infty)$ as well. Therefore, the support of (U_1, U_2) is

$$R_{U_1,U_2} = \{(u_1, u_2) : -\infty < u_1 < \infty, \ -\infty < u_2 < \infty\} = \mathbb{R}^2.$$

To verify the transformation above is one-to-one, we show $h(y_1, y_2) = h(y_1^*, y_2^*) \Longrightarrow y_1 = y_1^*$ and $y_2 = y_2^*$, where

$$h\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} \frac{y_1}{y_2} \\ y_2 \end{pmatrix}.$$

Suppose $h(y_1, y_2) = h(y_1^*, y_2^*)$. The second equation immediately implies $y_2 = y_2^*$. Plugging this into the first equation implies $y_1 = y_1^*$. Therefore, the transformation is 1:1.

The inverse transformation is found by solving

$$\begin{aligned} u_1 &= \frac{y_1}{y_2} \\ u_2 &= y_2. \end{aligned}$$

for $y_1 = h_1^{-1}(u_1, u_2)$ and $y_2 = h_2^{-1}(u_1, u_2)$. The second equation implies $y_2 = u_2$, so the first equation becomes

$$u_1 = \frac{y_1}{u_2} \implies y_1 = u_1 u_2.$$

Summarizing, we have

$$\begin{aligned} y_1 = h_1^{-1}(u_1, u_2) &= u_1 u_2 \\ y_2 = h_2^{-1}(u_1, u_2) &= u_2. \end{aligned}$$

The Jacobian is

$$J = \det \begin{vmatrix} \dfrac{\partial h_1^{-1}(u_1, u_2)}{\partial u_1} & \dfrac{\partial h_1^{-1}(u_1, u_2)}{\partial u_2} \\ \dfrac{\partial h_2^{-1}(u_1, u_2)}{\partial u_1} & \dfrac{\partial h_2^{-1}(u_1, u_2)}{\partial u_2} \end{vmatrix} = \det \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2(1) - u_1(0) = u_2.$$

Therefore, the joint pdf of $\mathbf{U} = (U_1, U_2)$, where nonzero, is

$$\begin{aligned} f_{U_1,U_2}(u_1, u_2) &= f_{Y_1,Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))|J| \\ &= f_{Y_1,Y_2}(u_1 u_2, u_2)\,|u_2| \\ &= \frac{|u_2|}{2\pi} e^{-[(u_1 u_2)^2 + u_2^2]/2} = \frac{|u_2|}{2\pi} e^{-u_2^2(1+u_1^2)/2}. \end{aligned}$$

Summarizing,

$$f_{U_1,U_2}(u_1,u_2) = \begin{cases} \dfrac{|u_2|}{2\pi} e^{-u_2^2(1+u_1^2)/2}, & -\infty < u_1 < \infty, \ -\infty < u_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

We are done performing the bivariate transformation. Now, to find the marginal pdf of $U_1 = Y_1/Y_2$, we integrate $f_{U_1,U_2}(u_1,u_2)$ over u_2. To do this, first we write

$$f_{U_1,U_2}(u_1,u_2) = \frac{|u_2|}{2\pi} e^{-u_2^2(1+u_1^2)/2} = \frac{|u_2|}{2\pi} e^{-u_2^2/\beta},$$

where

$$\beta = \frac{2}{1+u_1^2}.$$

For $-\infty < u_1 < \infty$, the marginal pdf of U_1 is

$$f_{U_1}(u_1) = \int_{-\infty}^{\infty} f_{U_1,U_2}(u_1,u_2)du_2 = \int_{-\infty}^{\infty} \underbrace{\frac{|u_2|}{2\pi} e^{-u_2^2/\beta}}_{=\ g(u_2)} du_2,$$

where the function $g(u_2) = \frac{|u_2|}{2\pi} e^{-u_2^2/\beta}$. Note that $g(-u_2) = g(u_2)$ for all $-\infty < u_2 < \infty$; i.e., g is a nonnegative even function and therefore $g(u_2)$ is symmetric about 0. Therefore,

$$\int_{-\infty}^{\infty} \frac{|u_2|}{2\pi} e^{-u_2^2/\beta} du_2 = 2\int_0^{\infty} \frac{u_2}{2\pi} e^{-u_2^2/\beta} du_2 = \frac{1}{\pi}\int_0^{\infty} u_2 e^{-u_2^2/\beta} du_2.$$

In the last integral, let

$$v = u_2^2 \implies dv = 2u_2 du_2.$$

The limits do not change under this transformation; as $u_2 : 0 \to \infty$, we have $v : 0 \to \infty$. Therefore,

$$\begin{aligned} f_{U_1}(u_1) = \frac{1}{\pi}\int_0^{\infty} u_2 e^{-u_2^2/\beta} du_2 &= \frac{1}{\pi}\int_0^{\infty} u_2 e^{-v/\beta} \frac{dv}{2u_2} \\ &= \frac{1}{2\pi}\int_0^{\infty} e^{-v/\beta} dv = \frac{\beta}{2\pi} \underbrace{\int_0^{\infty} \frac{1}{\beta} e^{-v/\beta} dv}_{=\ 1}, \end{aligned}$$

because $\frac{1}{\beta}e^{-v/\beta}$ is the exponential(β) pdf and we are integrating over $(0,\infty)$. Finally, we have

$$f_{U_1}(u_1) = \frac{\beta}{2\pi} = \frac{1}{2\pi}\left(\frac{2}{1+u_1^2}\right) = \frac{1}{\pi(1+u_1^2)}.$$

Summarizing,

$$f_{U_1}(u_1) = \begin{cases} \dfrac{1}{\pi(1+u_1^2)}, & -\infty < u_1 < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

We recognize this as the standard Cauchy pdf. Therefore, we have shown

$$Y_1 \sim \mathcal{N}(0,1),\ Y_2 \sim \mathcal{N}(0,1),\ Y_1 \perp\!\!\!\perp Y_2 \implies \frac{Y_1}{Y_2} \sim \text{(standard) Cauchy}.$$

6.105. The pdf of Y is

$$f_Y(y) = \begin{cases} \dfrac{y^{\alpha-1}}{B(\alpha,\beta)(1+y)^{\alpha+\beta}}, & y > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where recall

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

is the beta function. To find the pdf of $U = h(Y) = 1/(1+Y)$, we will use the transformation method. Note that

$$y > 0 \implies 1 + y > 1 \implies u = \frac{1}{1+y} \in (0,1).$$

Therefore, the support of U is $R_U = \{u : 0 < u < 1\}$. Note $u = h(y) = 1/(1+y)$ is a monotone decreasing function over $(0,\infty)$ because

$$\frac{d}{dy}h(y) = \frac{d}{dy}\left(\frac{1}{1+y}\right) = (-1)(1+y)^{-2} < 0$$

for all $y > 0$. Therefore, $h(y) = 1/(1+y)$ is one-to-one and we can use the transformation method. The inverse transformation is found as follows:

$$u = h(y) = \frac{1}{1+y} \implies 1 + y = \frac{1}{u} \implies y = h^{-1}(u) = \frac{1}{u} - 1.$$

The derivative of the inverse transformation is

$$\frac{d}{du}h^{-1}(u) = \frac{d}{du}\left(\frac{1}{u} - 1\right) = -\frac{1}{u^2}.$$

Therefore, for $0 < u < 1$, the pdf of U is

$$\begin{aligned}
f_U(u) &= f_Y(h^{-1}(u))\left|\frac{d}{du}h^{-1}(u)\right| \\
&= \frac{\left(\frac{1}{u}-1\right)^{\alpha-1}}{B(\alpha,\beta)\left[1+\left(\frac{1}{u}-1\right)\right]^{\alpha+\beta}}\left|-\frac{1}{u^2}\right| \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{1-u}{u}\right)^{\alpha-1}u^{\alpha+\beta}\times\frac{1}{u^2} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-u)^{\alpha-1}u^{\alpha+\beta-\alpha+1-2} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}u^{\beta-1}(1-u)^{\alpha-1}.
\end{aligned}$$

Summarizing, the pdf of $U = h(Y) = 1/(1+Y)$ is

$$f_U(u) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}u^{\beta-1}(1-u)^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

We recognize this as our usual beta distribution with parameters β and α.

6.107. We derived a general expression for the pdf of $U = Y^2$ in the notes; see pp 6-7. The pdf of U, where nonzero, is

$$f_U(u) = \frac{1}{2\sqrt{u}}\left[f_Y(\sqrt{u}) + f_Y(-\sqrt{u})\right].$$

If $Y \sim \mathcal{U}(-1, 3)$, then the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{1}{4}, & -1 < y < 3 \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$-1 < y < 3 \implies 0 < u < 9.$$

Therefore, the support of U is $R_U = \{u : 0 < u < 9\}$. To find the pdf of U, we have to consider the following two cases:

Case 1: $-1 < y \leq 1 \iff 0 < u \leq 1$. We have

$$\begin{aligned} f_Y(\sqrt{u}) &= \frac{1}{4} \\ f_Y(-\sqrt{u}) &= \frac{1}{4}. \end{aligned}$$

Therefore,

$$f_U(u) = \frac{1}{2\sqrt{u}}\left[f_Y(\sqrt{u}) + f_Y(-\sqrt{u})\right] = \frac{1}{4\sqrt{u}}.$$

Case 2: $1 < y < 3 \iff 1 < u < 9$. We have

$$\begin{aligned} f_Y(\sqrt{u}) &= \frac{1}{4} \\ f_Y(-\sqrt{u}) &= 0. \end{aligned}$$

Therefore,

$$f_U(u) = \frac{1}{2\sqrt{u}}\left[f_Y(\sqrt{u}) + f_Y(-\sqrt{u})\right] = \frac{1}{8\sqrt{u}}.$$

Combining both cases, the pdf of $U = Y^2$ is

$$f_U(u) = \begin{cases} \frac{1}{4\sqrt{u}}, & 0 < u \leq 1 \\ \frac{1}{8\sqrt{u}}, & 1 < u < 9 \\ 0, & \text{otherwise.} \end{cases}$$