7.11. In this problem, we envision the sample $Y_1, Y_2, ..., Y_9$, where

$$Y_i = \text{basal area of } i\text{th tree (measured in sq inches)}, \quad i = 1, 2, ..., 9.$$

We assume the population distribution is $\mathcal{N}(\mu, 16)$, and μ is the population mean basal area. We regard $Y_1, Y_2, ..., Y_9$ as an iid sample from the $\mathcal{N}(\mu, 16)$ population distribution. We want to find

$$P(-2 < \overline{Y} - \mu < 2).$$

Note that

$$\overline{Y} = \frac{1}{9}\sum_{i=1}^{9} Y_i$$

is the sample mean basal area and μ is the population mean. The difference between them is $\overline{Y} - \mu$. We know the sampling distribution

$$Z = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} = \frac{\overline{Y} - \mu}{4/\sqrt{9}} \sim \mathcal{N}(0, 1).$$

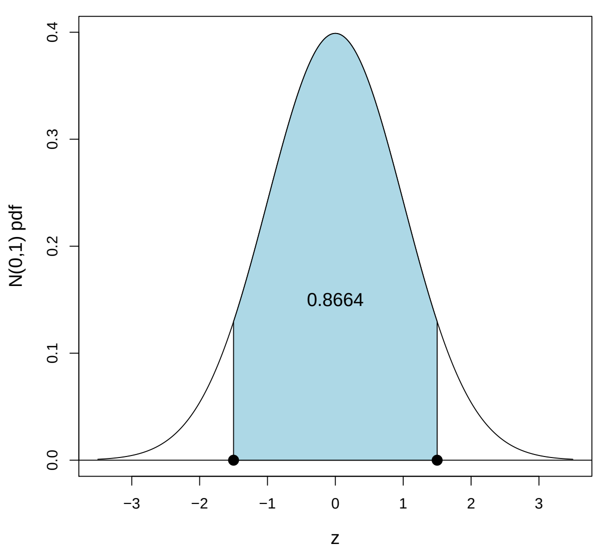
Therefore,

$$P(-2 < \overline{Y} - \mu < 2) = P\left(-\frac{2}{4/\sqrt{9}} < \frac{\overline{Y} - \mu}{4/\sqrt{9}} < \frac{2}{4/\sqrt{9}}\right) = P(-1.5 < Z < 1.5),$$

where $Z \sim \mathcal{N}(0, 1)$. This probability is easy to calculate in R:

```
> pnorm(1.5,0,1)-pnorm(-1.5,0,1)   #P(-1.5 < Z < 1.5)
[1] 0.8663856
```

See the $\mathcal{N}(0, 1)$ pdf shown below:



7.13. In this problem, we envision the sample $Y_1, Y_2, ..., Y_{10}$, where

$$Y_i = \ln(\text{LC50}) \text{ concentration for } i\text{th study (measured in mg/l)}, \quad i = 1, 2, ..., 10.$$

We assume the population distribution is $\mathcal{N}(\mu, 0.4)$, and μ is the population mean ln(LC50) concentration. We regard $Y_1, Y_2, ..., Y_{10}$ as an iid sample from the $\mathcal{N}(\mu, 0.4)$ population distribution. We want to find

$$P(-0.5 < \overline{Y} - \mu < 0.5).$$

Note that

$$\overline{Y} = \frac{1}{10}\sum_{i=1}^{10} Y_i$$

is the sample mean ln(LC50) concentration and μ is the population mean. The difference between them is $\overline{Y} - \mu$. We know the sampling distribution

$$Z = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} = \frac{\overline{Y} - \mu}{\sqrt{0.4}/\sqrt{10}} \sim \mathcal{N}(0, 1).$$

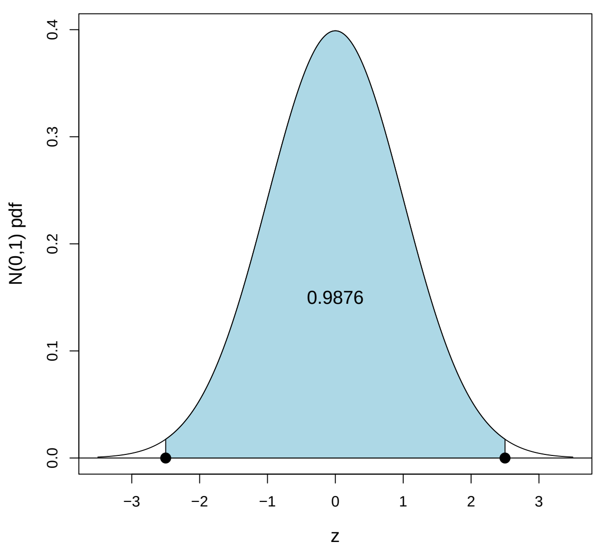
Therefore,

$$P(-0.5 < \overline{Y} - \mu < 0.5) = P\left(-\frac{0.5}{\sqrt{0.4}/\sqrt{10}} < \frac{\overline{Y} - \mu}{\sqrt{0.4}/\sqrt{10}} < \frac{0.5}{\sqrt{0.4}/\sqrt{10}}\right) = P(-2.5 < Z < 2.5),$$

where $Z \sim \mathcal{N}(0, 1)$. This probability is easy to calculate in R:

```
> pnorm(2.5,0,1)-pnorm(-2.5,0,1)   #P(-2.5 < Z < 2.5)
[1] 0.9875807
```

See the $\mathcal{N}(0, 1)$ pdf shown below:



7.19. In this problem, we envision the sample $Y_1, Y_2, ..., Y_{10}$, where

$$Y_i = i\text{th gauge reading (measured in amps)}, \quad i = 1, 2, ..., 10.$$

We assume the population distribution is $\mathcal{N}(\mu, \sigma^2)$, where μ is the population mean reading and σ^2 is the population variance. We regard $Y_1, Y_2, ..., Y_{10}$ as an iid sample from the $\mathcal{N}(\mu, \sigma^2)$ population distribution.

The manufacturer markets the ammeters to have a population standard deviation no larger than $\sigma = 0.2$ amps$-$this means the population variance is no more than $\sigma^2 = 0.04$ (amps)^2. If the sample variance of $n = 10$ readings is $s^2 = 0.065$, we are being asked to determine if this is "unusual" given the manufacturer's claim that σ^2 is not larger than 0.04. To gain insight on this, we can calculate

$$P(S^2 > 0.065)$$

under the assumption that $\sigma^2 = 0.04$. If this probability is "small," then this might lead us to suspect the manufacturer's claim. We know the sampling distribution

$$W = \frac{(n-1)S^2}{\sigma^2} = \frac{9S^2}{0.04} \sim \chi^2(9).$$

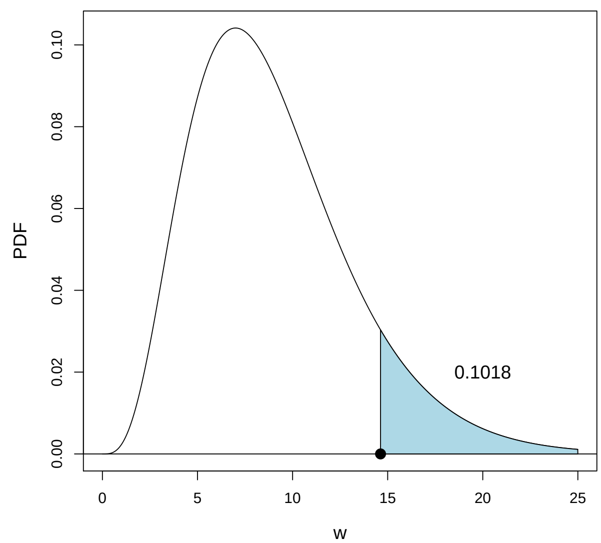
Therefore,

$$P(S^2 > 0.065) = P\left(\frac{9S^2}{0.04} > \frac{9(0.065)}{0.04}\right) = P\left(W > 14.625\right)$$

where $W \sim \chi^2(9)$. This probability is easy to calculate in R:

```
> 1-pchisq(14.625,9)   # P(W>14.625)
[1] 0.1017651
```

See the $\chi^2(9)$ pdf shown below:



The probability $P(S^2 > 0.065) \approx 0.1018$ is "small," but it might not be regarded as so small that it would cause us to seriously doubt the manufacturer's claim. If this probability was something like 0.0001, then that would be different. In this problem (for those of you that have had some applied statistics), you are essentially calculating a "p-value" for the test of

$$H_0 : \sigma^2 = 0.04$$
versus
$$H_1 : \sigma^2 > 0.04$$

by using the "test statistic"

$$W = \frac{9S^2}{0.04} \overset{H_0}{\sim} \chi^2(9).$$

7.29. We are given that $Y \sim F(\nu_1, \nu_2)$. Recall the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{y^{\frac{\nu_1}{2}-1}}{[1+(\frac{\nu_1}{\nu_2})y]^{\frac{\nu_1+\nu_2}{2}}}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

We want to find the pdf of

$$U = h(Y) = \frac{1}{Y}.$$

We will use the transformation method. Note that

$$y > 0 \implies u = \frac{1}{y} > 0.$$

Therefore, the support of U is $R_U = \{u : u > 0\}$. Note $u = h(y) = 1/y$ is a monotone decreasing function over $(0, \infty)$. Therefore, $h(y) = 1/y$ is one-to-one and we can use the transformation method. The inverse transformation is found as follows:

$$u = h(y) = \frac{1}{y} \implies y = h^{-1}(u) = \frac{1}{u}.$$

The derivative of the inverse transformation is

$$\frac{d}{du}h^{-1}(u) = \frac{d}{du}\left(\frac{1}{u}\right) = -\frac{1}{u^2}.$$

Therefore, for $u > 0$, the pdf of U is

$$\begin{aligned}
f_U(u) &= f_Y(h^{-1}(u)) \left|\frac{d}{du}h^{-1}(u)\right| \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{(\frac{1}{u})^{\frac{\nu_1}{2}-1}}{[1+(\frac{\nu_1}{\nu_2})(\frac{1}{u})]^{\frac{\nu_1+\nu_2}{2}}} \left|-\frac{1}{u^2}\right| \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{1}{u^{\frac{\nu_1}{2}+1}} \frac{1}{\left(\frac{u+\frac{\nu_1}{\nu_2}}{u}\right)^{\frac{\nu_1+\nu_2}{2}}} \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{u^{\frac{\nu_1+\nu_2}{2}}}{u^{\frac{\nu_1}{2}+1}} \frac{1}{\left(u+\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1+\nu_2}{2}}} \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} u^{\frac{\nu_2}{2}-1} \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{\left(u+\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}} \frac{1}{\left(u+\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_2}{2}}} \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} u^{\frac{\nu_2}{2}-1} \frac{1}{\left[1+(\frac{\nu_2}{\nu_1})u\right]^{\frac{\nu_1}{2}}} \frac{1}{\left(\frac{\nu_2 u+\nu_1}{\nu_2}\right)^{\frac{\nu_2}{2}}} \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} u^{\frac{\nu_2}{2}-1} \frac{1}{\left[1+(\frac{\nu_2}{\nu_1})u\right]^{\frac{\nu_1}{2}}} \left(\frac{\nu_2}{\nu_2 u+\nu_1}\right)^{\frac{\nu_2}{2}} \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \nu_2^{\frac{\nu_2}{2}} u^{\frac{\nu_2}{2}-1} \frac{1}{\left[1+(\frac{\nu_2}{\nu_1})u\right]^{\frac{\nu_1}{2}}} \frac{1}{\left(\frac{\nu_2 u+\nu_1}{\nu_1}\right)^{\frac{\nu_2}{2}} \nu_1^{\frac{\nu_2}{2}}} \\
&= \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}} u^{\frac{\nu_2}{2}-1} \frac{1}{\left[1+(\frac{\nu_2}{\nu_1})u\right]^{\frac{\nu_1}{2}}} \frac{1}{\left[1+(\frac{\nu_2}{\nu_1})u\right]^{\frac{\nu_2}{2}}} = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_2}{\nu_1}\right)^{\frac{\nu_2}{2}} \frac{u^{\frac{\nu_2}{2}-1}}{\left[1+(\frac{\nu_2}{\nu_1})u\right]^{\frac{\nu_1+\nu_2}{2}}}.
\end{aligned}$$

We recognize this as the $F(\nu_2, \nu_1)$ pdf; therefore, the result. In this problem, we have shown

$$Y \sim F(\nu_1, \nu_2) \implies U = \frac{1}{Y} \sim F(\nu_2, \nu_1).$$

7.33. In this problem, we are being asked to "verify" that

$$T \sim t(\nu) \implies U = T^2 \sim F(1, \nu).$$

To do this problem rigorously, we would do a transformation like we did in Problem 7.29. However, we would encounter a problem. Note that

$$-\infty < t < \infty \implies u = t^2 \geq 0.$$

However, the function $u = t^2$ is not 1:1 over $(-\infty, \infty)$. Therefore, we could not use our transformation result as stated; instead, we would have to first generalize our transformation technique to handle non-1:1 functions. If anyone wants to know how to do this (i.e., extend our transformation method to handle non-monotone transformations), then stop by my office and I will tell you.

Appealing to the definitions of t and F distributions makes this problem "heuristic" in nature. Suppose $Z \sim \mathcal{N}(0, 1)$, $W_2 \sim \chi^2(\nu)$, and $Z \perp\!\!\!\perp W_2$. We know

$$T = \frac{Z}{\sqrt{W_2/\nu}} \sim t(\nu).$$

Therefore,

$$T^2 = \left(\frac{Z}{\sqrt{W_2/\nu}}\right)^2 = \frac{Z^2}{W_2/\nu}.$$

We know $W_1 = Z^2 \sim \chi^2(1)$, so write

$$T^2 = \frac{W_1}{W_2/\nu} = \frac{W_1/1}{W_2/\nu} \sim F(1, \nu).$$

Note that $W_1 \perp\!\!\!\perp W_2$ because $Z \perp\!\!\!\perp W_2$ is true by assumption; i.e., W_1 is a function of Z, so it too is independent of W_2.

7.37. We are given $Y_1, Y_2, ..., Y_5$ are iid $\mathcal{N}(0, 1)$.
(a) Consider

$$W = \sum_{i=1}^{5} Y_i^2.$$

We know each $Y_i^2 \sim \chi^2(1)$, so

$$m_{Y_i^2}(t) = \left(\frac{1}{1-2t}\right)^{\frac{1}{2}},$$

for $i = 1, 2, ..., 5$. Now, W is the sum of 5 iid $\chi^2(1)$ random variables. The mgf of this sum is

$$m_W(t) = \left[\left(\frac{1}{1-2t}\right)^{\frac{1}{2}}\right]^5 = \left(\frac{1}{1-2t}\right)^{\frac{5}{2}},$$

which we recognize as the mgf of a $\chi^2(5)$ random variable. Because mgfs are unique, $W \sim \chi^2(5)$.

(b) Recall that $\sigma^2 = 1$. Therefore,

$$U = \sum_{i=1}^{5}(Y_i - \overline{Y})^2 = (5-1)S^2 = \frac{(5-1)S^2}{1} \sim \chi^2(4).$$

(c) We are given that $Y_6 \sim \mathcal{N}(0,1)$, which is independent of $Y_1, Y_2, ..., Y_5$. We know

$$U = \sum_{i=1}^{5}(Y_i - \overline{Y})^2 \sim \chi^2(4)$$

from part (b). Also, $Y_6^2 \sim \chi^2(1)$. Because $\sum_{i=1}^{5}(Y_i - \overline{Y})^2$ is independent of Y_6^2 (functions of independent random variables are also independent), the mgf of

$$V = \sum_{i=1}^{n}(Y_i - \overline{Y})^2 + Y_6^2$$

is

$$m_V(t) = \left(\frac{1}{1-2t}\right)^{\frac{4}{2}}\left(\frac{1}{1-2t}\right)^{\frac{1}{2}} = \left(\frac{1}{1-2t}\right)^{\frac{5}{2}}.$$

We recognize this as the mgf of a $\chi^2(5)$ random variable. Because mgfs are unique, $V \sim \chi^2(5)$.

7.38. This is a continuation of Problem 7.37.
(a) We know $Y_6 \sim \mathcal{N}(0,1)$ and $W \sim \chi^2(5)$. We also know $Y_6 \perp\!\!\!\perp W$ because W depends only on $Y_1, Y_2, ..., Y_5$. Therefore,

$$\frac{\sqrt{5}Y_6}{\sqrt{W}} = \frac{Y_6}{\sqrt{W/5}} \sim \frac{\text{“}\mathcal{N}(0,1)\text{”}}{\sqrt{\frac{\text{“}\chi^2(5)\text{”}}{5}}} \sim t(5).$$

(b) This is similar to part (a). We know $Y_6 \sim \mathcal{N}(0,1)$ and $U \sim \chi^2(4)$. We also know $Y_6 \perp\!\!\!\perp U$ because U depends only on $Y_1, Y_2, ..., Y_5$. Therefore,

$$\frac{2Y_6}{\sqrt{U}} = \frac{Y_6}{\sqrt{U/4}} \sim \frac{\text{“}\mathcal{N}(0,1)\text{”}}{\sqrt{\frac{\text{“}\chi^2(4)\text{”}}{4}}} \sim t(4).$$

(c) In the numerator, we have

$$\overline{Y} \sim \mathcal{N}\left(0, \frac{1}{5}\right).$$

Consider the random variable $\sqrt{5Y}$. Note that

$$\begin{aligned} E(\sqrt{5}\overline{Y}) &= \sqrt{5}E(\overline{Y}) = \sqrt{5}(0) = 0 \\ V(\sqrt{5}\overline{Y}) &= 5V(\overline{Y}) = \frac{5}{5} = 1. \end{aligned}$$

Also, $\sqrt{5}\overline{Y}$ is a linear function of $\overline{Y}$, which is normal. Therefore,

$$\sqrt{5}\overline{Y} \sim \mathcal{N}(0,1) \implies 5\overline{Y}^2 = (\sqrt{5}\overline{Y})^2 \sim \chi^2(1).$$

We already know $Y_6^2 \sim \chi^2(1)$. Because $5\overline{Y}^2$ is independent of Y_6^2 (functions of independent random variables are also independent),

$$5\overline{Y}^2 + Y_6^2 \sim \chi^2(2).$$

We know $U \sim \chi^2(4)$, so let's write

$$\frac{2(5\overline{Y}^2 + Y_6^2)}{U} = \frac{(5\overline{Y}^2 + Y_6^2)/2}{U/4} \sim \frac{\text{``}\chi^2(2)\text{''}/2}{\text{``}\chi^2(4)\text{''}/4} \sim F(2,4).$$

We must argue the numerator and denominator are independent. Note that $5\overline{Y}^2 \perp\!\!\!\perp U$ because $U = 4S^2$ and $\overline{Y} \perp\!\!\!\perp S^2$. Also, $Y_6^2 \perp\!\!\!\perp U$ because U depends only on $Y_1, Y_2, ..., Y_5$. Therefore, $(5\overline{Y}^2 + Y_6^2) \perp\!\!\!\perp U$ and we are done.

7.39. This problem examines different sampling distributions that arise in the analysis of variance (ANOVA) of one-way layouts. We have independent random samples

$$\begin{aligned} X_{11}, X_{12}, ..., X_{1n_1} &\sim \text{ iid } \mathcal{N}(\mu_1, \sigma^2) \quad \longleftarrow \text{ sample from ``treatment group 1''} \\ X_{21}, X_{22}, ..., X_{2n_2} &\sim \text{ iid } \mathcal{N}(\mu_2, \sigma^2) \quad \longleftarrow \text{ sample from ``treatment group 2''} \\ &\vdots \\ X_{k1}, X_{k2}, ..., X_{kn_k} &\sim \text{ iid } \mathcal{N}(\mu_k, \sigma^2) \quad \longleftarrow \text{ sample from ``treatment group } k\text{.''} \end{aligned}$$

Note that the population variance σ^2 is same in each of the k treatment group populations (a critical assumption in ANOVA).

(a) We know the sample mean $\overline{X}_i$ has the following sampling distribution:

$$\overline{X}_i \sim \mathcal{N}\left(\mu_i, \frac{\sigma^2}{n_i}\right), \quad i = 1, 2, ..., k.$$

Therefore, because

$$\widehat{\theta} = c_1\overline{X}_1 + c_2\overline{X}_2 + \cdots + c_k\overline{X}_k$$

is a linear combination of normal random variables, it is also normally distributed with mean

$$E(\widehat{\theta}) = E(c_1\overline{X}_1 + c_2\overline{X}_2 + \cdots + c_k\overline{X}_k) = c_1\mu_1 + c_2\mu_2 + \cdots + c_k\mu_k = \theta$$

and variance

$$V(\widehat{\theta}) = V(c_1\overline{X}_1 + c_2\overline{X}_2 + \cdots + c_k\overline{X}_k) = c_1^2\left(\frac{\sigma^2}{n_1}\right) + c_2^2\left(\frac{\sigma^2}{n_2}\right) + \cdots + c_k^2\left(\frac{\sigma^2}{n_k}\right) = \sigma^2\sum_{i=1}^{k}\frac{c_i^2}{n_i}.$$

The variance calculation follows because the sample means are independent so all the covariance terms are zero. We have shown

$$\widehat{\theta} \sim \mathcal{N}\left(\theta,\ \sigma^2\sum_{i=1}^{k}\frac{c_i^2}{n_i}\right).$$

(b) Let S_i^2 denote the sample variance of the ith sample, for $i = 1, 2, ..., k$. We know

$$\frac{(n_i - 1)S_i^2}{\sigma^2} \sim \chi^2(n_i - 1), \quad i = 1, 2, ..., k.$$

Because the samples are independent, we have

$$\frac{\text{SSE}}{\sigma^2} = \frac{\sum_{i=1}^k (n_i - 1)S_i^2}{\sigma^2} = \frac{(n_1 - 1)S_1^2}{\sigma^2} + \frac{(n_2 - 1)S_2^2}{\sigma^2} + \cdots + \frac{(n_k - 1)S_k^2}{\sigma^2} \sim \chi^2\left(\sum_{i=1}^k n_i - k\right);$$

i.e., the degrees of freedom "add" because of independence. **Note:** In the analysis of variance, SSE is called the error sum-of squares.

(c) From part (a), we know

$$\widehat{\theta} \sim \mathcal{N}\left(\theta,\ \sigma^2 \sum_{i=1}^k \frac{c_i^2}{n_i}\right) \quad \Longrightarrow \quad Z = \frac{\widehat{\theta} - \theta}{\sqrt{\sigma^2 \sum_{i=1}^k \frac{c_i^2}{n_i}}} \sim \mathcal{N}(0, 1).$$

Therefore,

$$\frac{\widehat{\theta} - \theta}{\sqrt{\left(\sum_{i=1}^k \frac{c_i^2}{n_i}\right)\text{MSE}}} = \frac{\dfrac{\widehat{\theta} - \theta}{\sqrt{\sigma^2 \sum_{i=1}^k \frac{c_i^2}{n_i}}}}{\sqrt{\dfrac{\text{SSE}}{\sigma^2} \Big/ \left(\sum_{i=1}^k n_i - k\right)}} \sim \frac{\text{"}\mathcal{N}(0,1)\text{"}}{\sqrt{\dfrac{\text{"}\chi^2(\sum_{i=1}^k n_i - k)\text{"}}{\sum_{i=1}^k n_i - k}}} \sim t\left(\sum_{i=1}^k n_i - k\right).$$

Note: In the analysis of variance, MSE is called the mean-squared error. The result in part (c) is used to write confidence intervals and perform hypothesis tests for linear combinations of population means in one-way layouts. If $\sum_{i=1}^k c_i = 1$, the linear combination $\theta = c_1\mu_1 + c_2\mu_2 + \cdots + c_k\mu_k$ is called a **contrast**.

7.88. In this problem, we envision the sample $Y_1, Y_2, ..., Y_8$, where

$$Y_i = \text{efficiency for } i\text{th bulb (measured in lumens/watt)}, \quad i = 1, 2, ..., 8.$$

We assume the population distribution is $\mathcal{N}(9.5, 0.5^2)$; i.e., $\mu = 9.5$ is the population mean efficiency and $\sigma^2 = 0.5^2$ is the population variance. We regard $Y_1, Y_2, ..., Y_8$ as an iid sample from the $\mathcal{N}(9.5, 0.5^2)$ population distribution. We want to find

$$P(\overline{Y} > 10).$$

Note that

$$\overline{Y} = \frac{1}{8}\sum_{i=1}^8 Y_i$$

is the sample mean efficiency. We know the sampling distribution of $\overline{Y}$ is

$$\overline{Y} \sim \mathcal{N}\left(9.5, \frac{0.5^2}{8}\right).$$

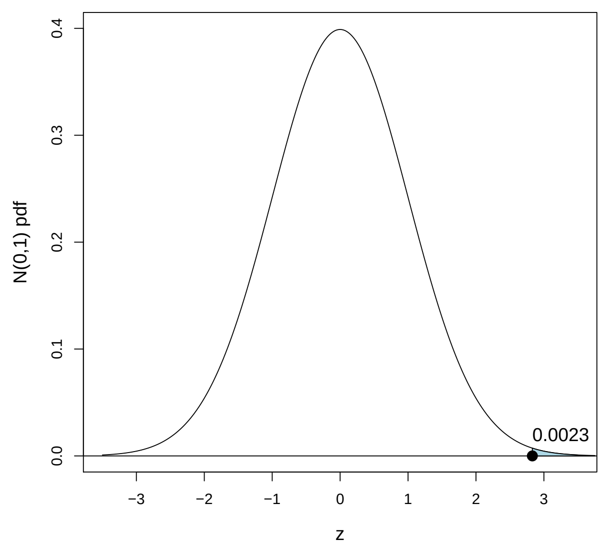
Therefore,

$$P(\overline{Y} > 10) = P\left(Z > \frac{10 - 9.5}{\sqrt{0.5^2/8}}\right) = P(Z > 2.83),$$

where $Z \sim \mathcal{N}(0,1)$. This probability is easy to calculate in R:

```
> 1-pnorm(2.83,0,1)  #P(Z>2.83)
[1] 0.0023274
```

See the $\mathcal{N}(0,1)$ pdf shown below:



Therefore, it is highly unlikely this specification for the room will be met, assuming the population distribution for bulb efficiency is $\mathcal{N}(9.5, 0.5^2)$.

7.95. This problem deals with a statistic called the **coefficient of variation**; i.e.,

$$T(\mathbf{Y}) = \frac{S}{\overline{Y}},$$

which is a measure of variability relative to the mean. This measure is useful if you want to compare the variation of two groups which may have drastically different means.

In this problem, $Y_1, Y_2, ..., Y_{10}$ are iid from a $\mathcal{N}(0, \sigma^2)$ population distribution; i.e., the population mean is $\mu = 0$ and the population variance is σ^2.

(a) We know

$$\overline{Y} \sim \mathcal{N}\left(0, \frac{\sigma^2}{10}\right) \implies Z = \frac{\overline{Y}}{\sigma/\sqrt{10}} \sim \mathcal{N}(0,1) \implies Z^2 = \left(\frac{\overline{Y}}{\sigma/\sqrt{10}}\right)^2 = \frac{10\overline{Y}}{\sigma^2} \sim \chi^2(1).$$

We also know

$$\frac{(n-1)S^2}{\sigma^2} = \frac{9S^2}{\sigma^2} \sim \chi^2(9).$$

Therefore, because

$$\frac{10\overline{Y}}{\sigma^2} \perp\!\!\!\perp \frac{9S^2}{\sigma^2},$$

we have

$$F = \frac{10\overline{Y}^2}{S^2} = \frac{\frac{10\overline{Y}}{\sigma^2}\Big/1}{\frac{9S^2}{\sigma^2}\Big/9} \sim \frac{\text{``}\chi^2(1)\text{''}/1}{\text{``}\chi^2(9)\text{''}/9} \sim F(1,9).$$

(b) From Problem 7.29, we know

$$\frac{1}{F} = \frac{S^2}{10\overline{Y}^2} \sim F(9,1).$$

(c) Suppose $c > 0$. We have

$$0.95 = P\left(-c < \frac{S}{\overline{\overline{Y}}} < c\right) = P\left(\frac{S^2}{\overline{Y}^2} < c^2\right) = P\left(\frac{S^2}{10\overline{Y}^2} < \frac{c^2}{10}\right).$$

For this equation to hold, it must be true that

$$\frac{c^2}{10} = F_{0.05,9,1};$$

i.e., $c^2/10$ is the 95th percentile of the $F(9,1)$ distribution. From R, we calculate this percentile to be

```
> qf(0.95,9,1)
[1] 240.5433
```

Therefore,

$$\frac{c^2}{10} \stackrel{\text{set}}{=} 240.5433 \implies c^2 = 2405.433 \implies c = \sqrt{2405.433} \approx 49.045.$$