STAT 520
FORECASTING AND TIME SERIES
2013 FALL
Homework 02

1.

2.7.(a) Since $\{Y_t\}$ is a stationary process, it has constant mean μ over time and autocovariance function γ_k which is free of t. Letting $W_t = \nabla Y_t = Y_t - Y_{t-1}$, we obtain the mean function of $\{W_t\}$

$$E(W_t) \quad = \quad E(Y_t - Y_{t-1}) = E(Y_t) - E(Y_{t-1}) = \mu - \mu = 0$$

which is constant over time t. The autocovariance function of $\{W_t\}$

$$\begin{aligned} & Cov(W_t, W_{t-k}) = Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\ = \ & Cov(Y_t, Y_{t-k}) + Cov(Y_t, -Y_{t-k-1}) + Cov(-Y_{t-1}, Y_{t-k}) + Cov(-Y_{t-1}, -Y_{t-k-1}) \\ = \ & Cov(Y_t, Y_{t-k}) - Cov(Y_t, Y_{t-k-1}) - Cov(Y_{t-1}, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-k-1}) \\ = \ & \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k = -\gamma_{k+1} + 2\gamma_k - \gamma_{k-1} \end{aligned}$$

only depends on time lag k. So we conclude that $\{W_t\}$ is stationary.

2.7.(b) From previous result, we know that the first difference of a stationary process is still stationary. In this case, since we proved that $\{W_t\}$ is stationary, then the first differences of $\{W_t\}$, $\{U_t\}$, are stationary as well.

2.9.(a) Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k and β_0 and β_1 are constant. The mean function of $\{Y_t\}$

$$E(Y_t) = E(\beta_0 + \beta_1 t + X_t) = \beta_0 + \beta_1 t + E(X_t) = \beta_0 + \beta_1 t + 0 = \beta_0 + \beta_1 t$$

which is not constant over time t. Then $\{Y_t\}$ is not stationary.

Consider $W_t = Y_t - Y_{t-1}$, we obtain the mean function of $\{W_t\}$

$$\begin{aligned} & E(W_t) = E(Y_t - Y_{t-1}) = E(Y_t) - E(Y_{t-1}) \\ = \ & E(\beta_0 + \beta_1 t + X_t) - E(\beta_0 + \beta_1 (t-1) + X_{t-1}) \\ = \ & \beta_0 + \beta_1 t + E(X_t) - \beta_0 + \beta_1 (t-1) + E(X_{t-1}) \\ = \ & \beta_0 + \beta_1 t + 0 - \beta_0 + \beta_1 (t-1) + 0 = \beta_1 \end{aligned}$$

which is constant over time t.

The autocovariance function of $\{W_t\}$

$$
\begin{aligned}
& Cov(W_t, W_{t-k}) = Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\
= \;& Cov\{\beta_0 + \beta_1 t + X_t - [\beta_0 + \beta_1(t-1) + X_{t-1}], \\
& \quad \beta_0 + \beta_1(t-k) + X_{t-k} - [\beta_0 + \beta_1(t-k-1) + X_{t-k-1}]\} \\
= \;& Cov(X_t - X_{t-1}, X_{t-k} - X_{t-k-1}) \\
= \;& Cov(X_t, X_{t-k}) + Cov(X_t, -X_{t-k-1}) + Cov(-X_{t-1}, X_{t-k}) + Cov(-X_{t-1}, -X_{t-k-1}) \\
= \;& Cov(X_t, X_{t-k}) - Cov(X_t, X_{t-k-1}) - Cov(X_{t-1}, X_{t-k}) + Cov(X_{t-1}, X_{t-k-1}) \\
= \;& \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k = -\gamma_{k+1} + 2\gamma_k - \gamma_{k-1}
\end{aligned}
$$

only depends on time lag k. We conclude that $\{W_t\}$ is stationary.

2.10. Let $\{X_t\}$ be a zero-mean, unit-variance stationary process with autocorrelation function ρ_k. Suppose that μ_t is a nonconstant function and that σ_t is a positive-valued nonconstant function. The observed series is formed as $Y_t = \mu_t + \sigma_t X_t$.

2.10.(a) Here we obtain the mean function

$$E(Y_t) \;=\; E(\mu_t + \sigma_t X_t) = \mu_t + \sigma_t E(X_t) = \mu_t + 0 = \mu_t$$

and the autocovariance function

$$
\begin{aligned}
Cov(Y_t, Y_{t-k}) &= Cov(\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k}) \\
&= Cov(\sigma_t X_t, \sigma_{t-k} X_{t-k}) = \sigma_t \sigma_{t-k} Cov(X_t, X_{t-k}) \\
&= \sigma_t \sigma_{t-k} \left(Corr(X_t, X_{t-k}) \sqrt{Var(X_t)\; Var(X_{t-k})} \right) \\
&= \sigma_t \sigma_{t-k} \left(\rho_k \sqrt{(1)(1)} \right) = \sigma_t \sigma_{t-k} \rho_k.
\end{aligned}
$$

2.10.(b) It is easy to obtain $Var(Y_t) = Var(\mu_t + \sigma_t X_t) = Var(\sigma_t X_t) = \sigma_t^2 Var(X_t) = \sigma_t^2(1) = \sigma_t^2$, then the autocorrelation function

$$
\begin{aligned}
Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)\; Var(Y_{t-k})}} \\
&= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{\sigma_t^2 \sigma_{t-k}^2}} = \frac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}} = \rho_k
\end{aligned}
$$

only depends on time lag k. However, $\{Y_t\}$ is not stationary because the mean function μ_t is not constant over time t.

2.10.(c) If we change the setting such that $\mu_t = \mu$ which is constant over time t, then $W_t = \mu + \sigma_t X_t$ has constant mean function μ and autocovariance function $Cov(W_t, W_{t-k}) = \sigma_t \sigma_{t-k} \rho_k$ by similar calculation above. Since the autocovariance function depends on time t, then $\{W_t\}$ is not stationary. However, it has constant mean μ over time t.

2.13 Let $Y_t = e_t - \theta(e_{t-1})^2$, where e_ts are independent and identically distributed (IID) random variables which follow $\mathcal{N}(0, \sigma_e^2)$ distribution. Then e_t^2/σ_e^2 are IID random variables which follow chi-squared distribution with 1 degree of freedom with $E(e_t^2/\sigma_e^2) = 1$ and $Var(e_t^2/\sigma_e^2) = 2$. In other words, $E(e_t^2) = \sigma_e^2$ and $Var(e_t^2) = 2\sigma_e^4$.

2.13.(a) First we obtain the autocovariance function

$$\begin{aligned} & Cov(Y_t, Y_{t-k}) = Cov(e_t - \theta(e_{t-1})^2, e_{t-k} - \theta(e_{t-k-1})^2) \\ = \; & Cov(e_t, e_{t-k}) + Cov(e_t, -\theta(e_{t-k-1})^2) + Cov(-\theta(e_{t-1})^2, e_{t-k}) \\ & + Cov(-\theta(e_{t-1})^2, -\theta(e_{t-k-1})^2) \\ = \; & Cov(e_t, e_{t-k}) + 0 + Cov(-\theta(e_{t-1})^2, e_{t-k}) + Cov(-\theta(e_{t-1})^2, -\theta(e_{t-k-1})^2) \\ = \; & Cov(e_t, e_{t-k}) - \theta Cov((e_{t-1})^2, e_{t-k}) + \theta^2 Cov((e_{t-1})^2, (e_{t-k-1})^2). \end{aligned}$$

If $k = 0$,

$$\begin{aligned} Cov(Y_t, Y_t) &= Cov(e_t, e_t) - \theta Cov((e_{t-1})^2, e_t) + \theta^2 Cov((e_{t-1})^2, (e_{t-1})^2) \\ &= Var(e_t) - 0 + \theta^2 Var(e_{t-1}^2) = \sigma_e^2 + \theta^2(2\sigma_e^4) = \sigma_e^2 + 2\theta^2\sigma_e^4. \end{aligned}$$

If $k = 1$,

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t, e_{t-1}) - \theta Cov((e_{t-1})^2, e_{t-1}) + \theta^2 Cov((e_{t-1})^2, (e_{t-2})^2) \\ &= 0 - \theta E\{[(e_{t-1})^2 - E((e_{t-1})^2)][e_{t-1} - E(e_{t-1})]\} + 0 \\ &= -\theta E\{[(e_{t-1})^2 - 1][e_t - 0]\} \\ &= -\theta E\{(e_t)^3 - e_t\} = -\theta E[(e_t)^3] + E(e_t) = 0 - 0 = 0. \end{aligned}$$

where $E[(e_t)^3] = 0$ because $\mathcal{N}(0, \sigma_e^2)$ distribution is symmetric.
If $k > 1$,

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t, e_{t-k}) - \theta Cov((e_{t-1})^2, e_{t-k}) + \theta^2 Cov((e_{t-1})^2, (e_{t-k-1})^2) \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

Then the autocovariance function

$$\gamma_k = Cov(Y_t, Y_{t-k}) \quad = \quad \begin{cases} \sigma_e^2 + 2\theta^2\sigma_e^4 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases}$$

which only depends on time lag k. And the autocorrelation function

$$\rho_k \quad = \quad \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases} .$$

2.13.(b) Here we obtain the mean function

$$E(Y_t) \quad = \quad E(e_t - \theta(e_{t-1})^2) = E(e_t) - \theta E((e_t)^2) = 0 - \theta = -\theta\sigma_e^2$$

which is free of time. Since the autocovariance function depends only on time lag k and the mean function is constant over time t, we conclude that $\{Y_t\}$ is stationary.

2.14.(a) For $Y_t = \theta_0 + te_t$, we have the mean function

$$E(Y_t) \quad = \quad E(\theta_0 + te_t) = \theta_0 + tE(e_t) = \theta_0 + 0 = \theta_0$$

which is free of time. And the autocovariance function

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(\theta_0 + te_t, \theta_0 + (t-k)e_{t-k}) = Cov(te_t, (t-k)e_{t-k}) \\ &= t(t-k)Cov(e_t, e_{t-k}) \\ &= \begin{cases} t^2\sigma_e^2 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases} \end{aligned}$$

which depends on time t. Then we conclude that $\{Y_t\}$ is stationary.

2.14.(b) For $W_t = \nabla Y_t = Y_t - Y_{t-1}$, first we expand W_t as

$$\begin{aligned} W_t &= \nabla Y_t = Y_t - Y_{t-1} = \theta_0 + te_t - (\theta_0 + (t-1)e_{t-1}) \\ &= te_t - (t-1)e_{t-1} \end{aligned}$$

and we have the mean function

$$E(W_t) \quad = \quad E(te_t - (t-1)e_{t-1}) = tE(e_t) - (t-1)E(e_{t-1}) = 0 - 0 = 0$$

which is free of time t. And the autocovariance function

$$\begin{aligned}
& Cov(W_t, W_{t-k}) = Cov(te_t - (t-1)e_{t-1}, (t-k)e_{t-k} - (t-k-1)e_{t-k-1}) \\
= \; & Cov(te_t, (t-k)e_{t-k}) + Cov(te_t, -(t-k-1)e_{t-k-1}) \\
& +Cov(-(t-1)e_{t-1}, (t-k)e_{t-k}) + Cov(-(t-1)e_{t-1}, -(t-k-1)e_{t-k-1}) \\
= \; & t(t-k)Cov(e_t, e_{t-k}) - t(t-k-1)Cov(e_t, e_{t-k-1}) \\
& -(t-1)(t-k)Cov(e_{t-1}, e_{t-k}) + (t-1)(t-k-1)Cov(e_{t-1}, e_{t-k-1}).
\end{aligned}$$

If $k = 0$,

$$\begin{aligned}
& Cov(W_t, W_t) \\
= \; & t(t)Cov(e_t, e_t) - t(t-1)Cov(e_t, e_{t-1}) \\
& -(t-1)(t)Cov(e_{t-1}, e_t) + (t-1)(t-1)Cov(e_{t-1}, e_{t-1}) \\
= \; & t^2 Var(e_t) - 0 - 0 + (t-1)^2 Var(e_{t-1}) \\
= \; & t^2\sigma_e^2 + (t-1)^2\sigma_e^2 = (2t^2 - 2t + 1)\sigma_e^2.
\end{aligned}$$

If $k = 1$,

$$\begin{aligned}
& Cov(W_t, W_{t-1}) \\
= \; & t(t-1)Cov(e_t, e_{t-1}) - t(t-2)Cov(e_t, e_{t-2}) \\
& -(t-1)(t-1)Cov(e_{t-1}, e_{t-1}) + (t-1)(t-2)Cov(e_{t-1}, e_{t-2}) \\
= \; & 0 - 0 - (t-1)^2 Var(e_{t-1}) + 0 = -(t-1)^2\sigma_e^2.
\end{aligned}$$

If $k > 1$,

$$\begin{aligned}
& Cov(W_t, W_{t-k}) \\
= \; & t(t-k)Cov(e_t, e_{t-k}) - t(t-k-1)Cov(e_t, e_{t-k-1}) \\
& -(t-1)(t-k)Cov(e_{t-1}, e_{t-k}) + (t-1)(t-k-1)Cov(e_{t-1}, e_{t-k-1}) \\
= \; & 0 - 0 - 0 + 0 = 0.
\end{aligned}$$

Then we obtain the autocovariance function

$$Cov(W_t, W_{t-k}) = \begin{cases} (2t^2 - 2t + 1)\sigma_e^2 & \text{for } k = 0 \\ -(t-1)^2\sigma_e^2 & \text{for } k = 1 \\ 0 & \text{for } k > 1 \end{cases}$$

which depends on time t. So we conclude that $\{W_t\}$ is not stationary.

2.14.(c) For $Y_t = e_t e_{t-1}$, we have the mean function

$$E(Y_t) = E(e_t e_{t-1}) = E(e_t)E(e_{t-1}) = 0(0) = 0$$

which is free of time t.

And the autocovariance function

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(e_t e_{t-1}, e_{t-k} e_{t-k-1}) \\ &= E(e_t e_{t-1} e_{t-k} e_{t-k-1}) - E(e_t e_{t-1}) E(e_{t-k} e_{t-k-1}). \end{aligned}$$

If $k = 0$,

$$\begin{aligned} Cov(Y_t, Y_t) &= E(e_t e_{t-1} e_t e_{t-1}) - E(e_t e_{t-1}) E(e_t e_{t-1}) \\ &= E(e_t^2 e_{t-1}^2) - [E(e_t e_{t-1})]^2 \\ &= E(e_t^2) E(e_{t-1}^2) - [E(e_t e_{t-1})]^2 \\ &= \sigma_e^2 \sigma_e^2 - [E(e_t) E(e_{t-1})]^2 = \sigma_e^4 - 0^2 \\ &= \sigma_e^4. \end{aligned}$$

since $\sigma_e^2 = Var(e_t) = E(e_t^2) - [E(e_t)]^2 = E(e_t^2) - 0 = E(e_t^2)$ for all time t.

If $k = 1$,

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= E(e_t e_{t-1} e_{t-1} e_{t-2}) - E(e_t e_{t-1}) E(e_{t-1} e_{t-2}) \\ &= E(e_t e_{t-1}^2 e_{t-2}) - E(e_t e_{t-1}) E(e_{t-1} e_{t-2}) \\ &= E(e_t) E(e_{t-1}^2) E(e_{t-2}) - E(e_t) E(e_{t-1}) E(e_{t-1}) E(e_{t-2}) \\ &= 0(\sigma_e^2)(0) - 0(0)(0)(0) = 0 - 0 \\ &= 0. \end{aligned}$$

If $k > 0$,

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= E(e_t e_{t-1} e_{t-k} e_{t-k-1}) - E(e_t e_{t-1}) E(e_{t-k} e_{t-k-1}) \\ &= E(e_t) E(e_{t-1}) E(e_{t-k}) E(e_{t-k-1}) - E(e_t) E(t e_{t-1}) E(e_{t-k}) E(e_{t-k-1}) \\ &= 0. \end{aligned}$$

Then we obtain the autocovariance function

$$Cov(Y_t, Y_{t-k}) = \begin{cases} \sigma_e^4 & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases}$$

which only depends on time lag k.

From previous results, we know that $\{Y_t\}$ has constant mean function and autocovariance function which only depends on time lag k. Then we conclude that $\{Y_t\}$ is stationary.

2.19. Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$, define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here θ_0 is constant.

2.19.(a) Here we expand Y_t for three times,

$$\begin{aligned} Y_t &= \theta_0 + Y_{t-1} + e_t = \theta_0 + (\theta_0 + Y_{t-2} + e_{t-1}) + e_t \\ &= 2\theta_0 + Y_{t-2} + e_{t-1} + e_t = 2\theta_0 + (\theta_0 + Y_{t-3} + e_{t-2}) + e_{t-1} + e_t \\ &= 3\theta_0 + Y_{t-3} + e_{t-2} + e_{t-1} + e_t. \end{aligned}$$

Then we observe that after expanding t times, Y_t can be rewrite as

$$Y_t = t\theta_0 + e_t + \cdots + e_1.$$

2.19.(b) Here we obtain the mean function

$$E(Y_t) = E(t\theta_0 + e_t + \cdots e_1) = t\theta_0 + E(e_t) + \cdots + E(e_1) = t\theta_0 + 0 + \cdots + 0 = t\theta_0.$$

2.19.(c) The autocovariance function

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(t\theta_0 + e_t + \cdots e_1, (t-k)\theta_0 + e_{t-k} + \cdots e_1) \\ &= Cov(e_t + \cdots e_1, e_{t-k} + \cdots e_1). \end{aligned}$$

For $k = 0$,

$$\begin{aligned} Cov(Y_t, Y_t) &= Cov(e_t + \cdots e_1, e_t + \cdots e_1) = Var(e_t + \cdots e_1) \\ &= Var(e_t) + \cdots Var(e_1) = t\sigma_t. \end{aligned}$$

For $k \geq 1$,

$$\begin{aligned} & Cov(Y_t, Y_{t-k}) = Cov(e_t + \cdots e_1, e_{t-k} + \cdots e_1) \\ = \; & Cov((e_t + \cdots + e_{t-k+1}) + e_{t-k} + \cdots + e_1, e_{t-k} + \cdots + e_1) \\ = \; & Cov(e_t + \cdots + e_{t-k+1}, e_{t-k} + \cdots + e_1) + Cov(e_{t-k} + \cdots + e_1, e_{t-k} + \cdots + e_1) \\ = \; & 0 + Var(e_{t-k} + \cdots + e_1) = Var(e_{t-k}) + \cdots + Var(e_1) \\ = \; & (t-k)\sigma_e^2. \end{aligned}$$

So we obtain the autocovariance function

$$Cov(Y_t, Y_{t-k}) = (t-k)\sigma_e^2 \text{ for } k \geq 0$$

which depends on time t and time lag k. Then we conclude that $\{Y_t\}$ is not stationary.

2.(a) First, we obtain the mean function

$$\begin{aligned} E(Y_t) &= E(Z_1 \cos(wt) + Z_2 \sin(wt) + e_t) \\ &= E(Z_1) \cos(wt) + E(Z_2) \sin(wt) + E(e_t) \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

which is constant over time t.

Then we divide the autocovariance function into three parts

$$\begin{aligned} & Cov(Y_t, Y_{t-k}) \\ = & Cov(Z_1 \cos(wt) + Z_2 \sin(wt) + e_t, Z_1 \cos(w(t-k)) + Z_2 \sin(w(t-k)) + e_{t-k}) \\ = & Cov(Z_1 \cos(wt), Z_1 \cos(w(t-k)) + Z_2 \sin(w(t-k)) + e_{t-k}) \quad (1) \\ & + Cov(Z_2 \sin(wt), Z_1 \cos(w(t-k)) + Z_2 \sin(w(t-k)) + e_{t-k}) \quad (2) \\ & + Cov(e_t, Z_1 \cos(w(t-k)) + Z_2 \sin(w(t-k)) + e_{t-k}). \quad (3) \end{aligned}$$

For part (1),

$$\begin{aligned} & Cov(Z_1 \cos(wt), Z_1 \cos(w(t-k)) + Z_2 \sin(w(t-k)) + e_{t-k}) \\ = & Cov(Z_1 \cos(wt), Z_1 \cos(w(t-k))) + Cov(Z_1 \cos(wt), Z_2 \sin(w(t-k))) \\ & + Cov(Z_1 \cos(wt), e_{t-k}) \\ = & \cos(wt) \cos(w(t-k)) Cov(Z_1, Z_1) + \cos(wt) \sin(w(t-k)) Cov(Z_1, Z_2) \\ & + \cos(wt) Cov(Z_1, e_{t-k}) \\ = & \cos(wt) \cos(w(t-k)) Var(Z_1) + 0 + 0 = \cos(wt) \cos(w(t-k)). \end{aligned}$$

For part (2),

$$\begin{aligned} & Cov(Z_2 \sin(wt), Z_1 \cos(w(t-k)) + Z_2 \sin(w(t-k)) + e_{t-k}) \\ = & Cov(Z_2 \sin(wt), Z_1 \cos(w(t-k))) + Cov(Z_2 \sin(wt), Z_2 \sin(w(t-k))) \\ & + Cov(Z_2 \sin(wt), +e_{t-k}) \\ = & \sin(wt) \cos(w(t-k)) Cov(Z_2, Z_1) + \sin(wt) \sin(w(t-k)) Cov(Z_2, Z_2) \\ & + \sin(wt) Cov(Z_2, +e_{t-k}) \\ = & 0 + \sin(wt) \sin(w(t-k)) Var(Z_2) + 0 = \sin(wt) \sin(w(t-k)). \end{aligned}$$

For part (3),

$$\begin{aligned} & Cov(e_t, Z_1 \cos(w(t-k)) + Z_2 \sin(w(t-k)) + e_{t-k}) \\ = & Cov(e_t, Z_1 \cos(w(t-k)) + Cov(e_t, Z_2 \sin(w(t-k))) + Cov(e_t, e_{t-k}) \\ = & 0 + 0 + Cov(e_t, e_{t-k}) = Cov(e_t, e_{t-k}). \end{aligned}$$

Then we combine the results of all three parts and obtain the autocovariance function

$$
\begin{aligned}
Cov(Y_t, Y_{t-k}) &= \cos(wt)\cos(w(t-k)) + \sin(wt)\sin(w(t-k)) + Cov(e_t, e_{t-k}) \\
&= \cos(wt - w(t-k)) = \cos(wk) + Cov(e_t, e_{t-k}) \\
&= \begin{cases} 1 + \sigma_e^2 & \text{for } k = 0 \\ \cos(wk) & \text{for } k \geq 1. \end{cases}
\end{aligned}
$$

According to calculations above, $\{Y_t\}$ has zero mean and autocovariance function which is only depends on k. Then we conclude that $\{Y_t\}$ is stationary.

2.(b)

Please see Figure 1. There is trend in this graph. The observations oscillate around zero between -4 to 4.

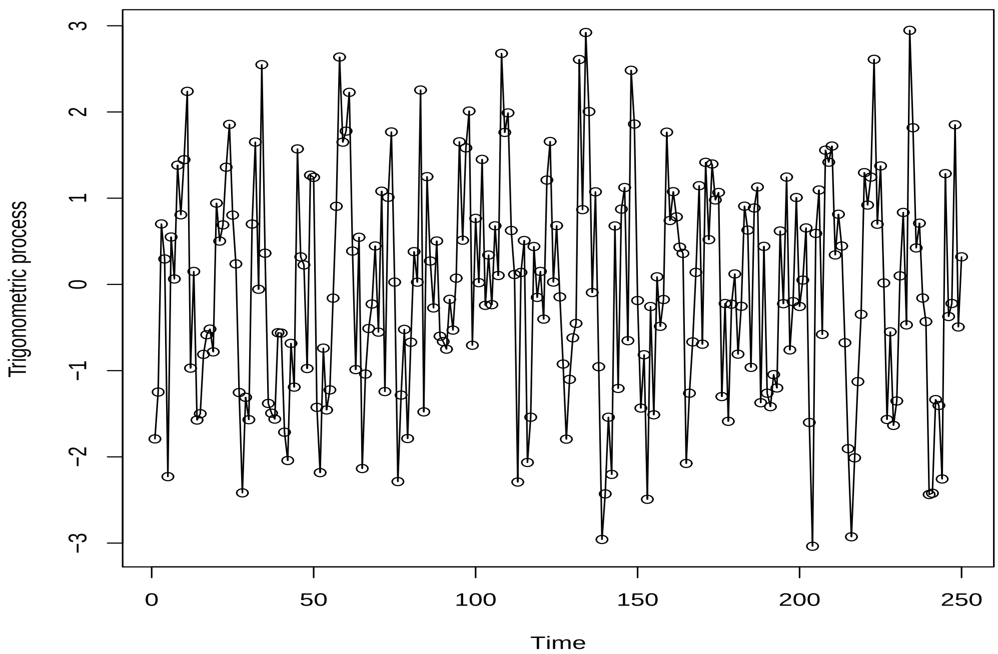

Figure 1: Time series plot for carbon dioxide data

The followings are `R` commands to make the graph above:

```
omega = 0.5
Z = rnorm(2,0.1)
e.t = rnorm(250,0,1)
Y.t = e.t*0
for (i in 1:length(e.t)){
     Y.t[i] = Z[1]*cos(omega*i) + Z[2]*sin(omega*i) + e.t[i]}
plot(Y.t,ylab="Trigonometric process", xlab="Time", type="o")
```

2.(c) There is an increasing trend of the time series plot because of adding the linear trend term $\beta_0 + \beta_1 t$ to model. So we can see that the mean function is not constant over time t and $\{Y_t\}$ appears not to be stationary.

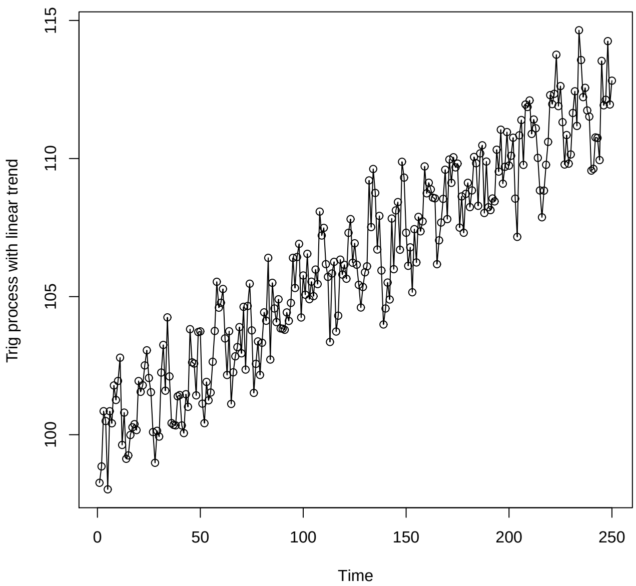

Figure 2: Time series plot for carbon dioxide data

The followings are R commands to make the graph above:

```
Y.tilde = e.t*0
for (i in 1:length(e.t)){
     Y.tilde[i] = 100 + 0.05*i + Z[1]*cos(omega*i) + Z[2]*sin
(omega*i) + e.t[i]}
```

```
plot(Y.tilde, ylab="Trig process with linear trend", xlab="Time", type="o")
```

2.(d) The first differences process $\{\nabla\tilde{Y}_t\}$ looks to have trend. From Problem **2.9**, we know that this process must be stationary.

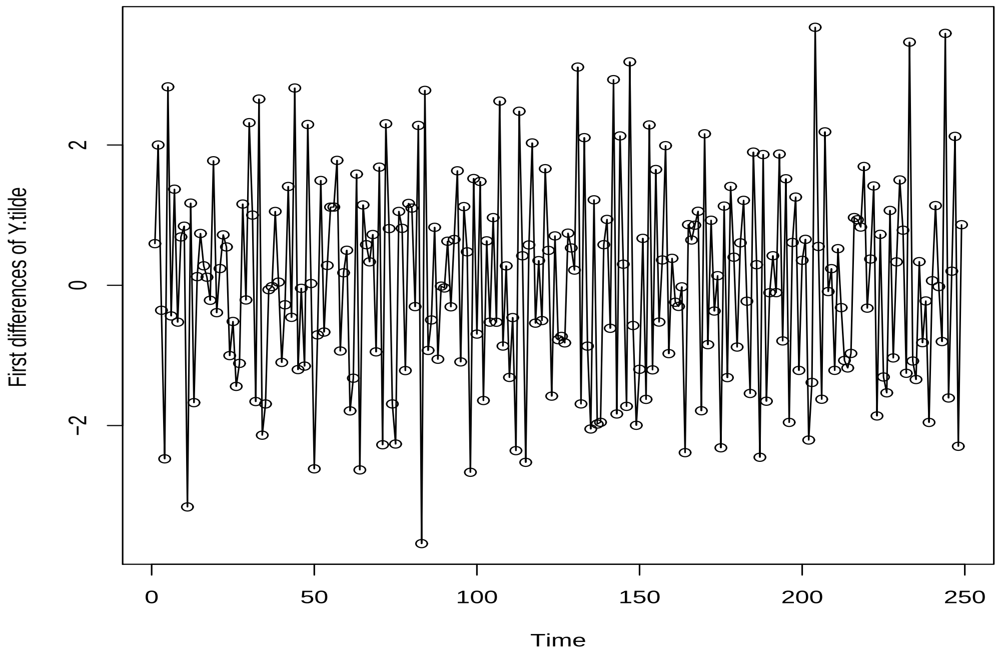

Figure 3: Time series plot for carbon dioxide data

The followings are R commands to make the graph above:

```
diff.Y.tilde = diff(Y.tilde)
plot(diff.Y.tilde, ylab="First differences of Y.tilde", xlab="Time", type="o")
```

3.(i) Here we give an example of a process $\{Y_t\}$ with constant mean but variance that increases with time. So we consider $Y_t = te_t$ with mean function

$$E(Y_t) \;=\; E(te_t) = tE(e_t) = t(0) = 0$$

which is constant. However, the variance

$$Var(Y_t) = Var(te_t) = t^2 Var(e_t) = t^2\sigma_e^2$$

increases with time.

3.(ii) Here we give an example of a stationary process whose autocovariance does not go to zero as time lag goes to infinity. So we consider $Y_t = X + e_t$ where X has mean zero and variance σ_X^2. X and white noise process $\{e_t\}$ are independent. Then we have the mean function of $\{Y_t\}$

$$E(Y_t) \;=\; E(X + e_t) = E(X) + E(e_t) = 0 + 0 = 0$$

which is constant over time t. The autocovariance function

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(X + e_t, X + e_{t-k}) \\ &= Cov(X, X) + Cov(X, e_{t-k}) + Cov(e_t, X) + Cov(e_t, e_{t-k}) \\ &= Var(X) + 0 + 0 + Cov(e_t, e_{t-k}) = Var(X) + Cov(e_t, e_{t-k}) \\ &= \begin{cases} \sigma_X^2 + \sigma_e^2 & \text{for } k = 0 \\ \sigma_X^2 & \text{for } k \geq 1 \end{cases} \end{aligned}$$

which only depends on time lag k. So we conclude that $\{Y_t\}$ is stationary. But the autocovariance function does not go to zero as time lag k goes to infinity.

3.(iii) Here we give an example of a nonstationary process $\{Y_t\}$ whose autocovariance depends only on time lag k. So we consider $Y_t = t + e_t$ with mean function

$$E(Y_t) \;=\; E(t + e_t) = t + E(e_t) = t + 0 = t$$

which is not free of time t. Then $\{Y_t\}$ is not stationary. Furthermore, we obtain the autocovariance function

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(t + e_t, (t-k) + e_{t-k}) = Cov(e_t, e_{t-k}) \\ &= \begin{cases} \sigma_e^2 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases} \end{aligned}$$

which depends only on time lag k.

3.(iv) Here we give an example of a process $\{Y_t\}$ that has nonzero autocorrelation only at lag $k = 1$. Consider $Y_t = e_t - e_{t-1}$ with mean function

$$E(Y_t) = E(e_t - e_{t-1}) = E(e_t) - E(e_{t-1}) = 0 - 0 = 0$$

which is free of time t. First, we obtain the autocovariance function

$$Cov(Y_t, Y_{t-k}) = \begin{cases} 2\sigma_e^2 & \text{for } k = 0 \\ -\sigma_e^2 & \text{for } k = 1 \\ 0 & \text{for } k \neq 1, k > 0 \end{cases},$$

which is free of time t. Then we conclude that $\{Y_t\}$ is stationary. (The calculation of autocorrelation function is similar to Problem **2.12.**. Please refer to Homework 1.) Then the autocorrelation function

$$\rho_k = \begin{cases} 1 & \text{for } k = 0 \\ -\frac{1}{2} & \text{for } k = 1 \\ 0 & \text{for } k \neq 1, k > 0 \end{cases}$$

has nonzero autocorrelation only at lag $k = 1$.

3.(v) Here we give an example of a nonstationary process $\{Y_t\}$ whose first differences are stationary. Consider the random process $\{Y_t\}$ which is given by **2.19.(c)** with $\theta_0 = 0$. That is $Y_t = e_t + \cdots + e_1$. By similar calculation, we know that $\{Y_t\}$ is not stationary. However, the first differences of $\{Y_t\}$, defined by $W_t = Y_t - Y_{t-1} = e_t$, is stationary since $\{e_t\}$ is a white noise process.