1. Suppose $X_1, X_2, ..., X_n$ is an iid sample from $f_X(x|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}$. Recall the score function is given by

$$S(\theta|\mathbf{x}) = \frac{\partial}{\partial\theta} \ln f_{\mathbf{X}}(\mathbf{x}|\theta),$$

where $f_{\mathbf{X}}(\mathbf{x}|\theta)$ is the joint distribution of $\mathbf{X} = (X_1, X_2, ..., X_n)$.
(a) Show that $E_\theta[S(\theta|\mathbf{X})] = 0$.
(b) Calculate the score function when

$$f_X(x|\theta) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{1+(x-\theta)^2}, & -\infty < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(c) How would you find the maximum likelihood estimate of $\theta$?

2. You have observed one observation $X$ from a distribution with probability density function $f_X(x)$ and support $\mathcal{X} = \{x : 0 < x < 1\}$.
(a) Derive the most powerful $\alpha = 0.05$ test for testing

$$H_0 : f_X(x) = 2x \; I(0 < x < 1)$$
versus
$$H_1 : f_X(x) = 5x^4 \; I(0 < x < 1).$$

Be sure to give the rejection region explicitly.
(b) Compute the power of the test.

3. Conditional on $\theta$, the random variables $X_1, X_2, ..., X_n$ are iid from

$$f_X(x|\theta) = \theta^2 x e^{-\theta x} I(x > 0).$$

In turn, the parameter $\theta$ is best regarded as random with prior distribution

$$\tau(\theta) = a e^{-a\theta} I(\theta > 0),$$

where $a > 0$ is known.
(a) Find the posterior mean of $\theta$.
(b) Discuss how you would formulate the Bayesian test of

$$H_0 : \theta \le \theta_0$$
versus
$$H_1 : \theta > \theta_0.$$

4. Suppose $X_1, X_2, ..., X_n$ are iid Bernoulli($\theta$), where $0 < \theta < 1$.
(a) Find the Crámer-Rao Lower Bound (CRLB) on the variance of any unbiased estimator of $\text{var}_\theta(X_1) = \theta(1-\theta)$.
(b) For $n \ge 3$, find the uniformly minimum variance unbiased estimator (UMVUE) of $\theta^3$.

5. Suppose $X_1, X_2, ..., X_n$ are iid Poisson random variables with mean $\theta > 0$.
(a) Show that the likelihood ratio test (LRT) of

$$H_0 : \theta \leq \theta_0$$
$$\text{versus}$$
$$H_1 : \theta > \theta_0$$

will reject $H_0$ if $\sum_{i=1}^{n} X_i > c'$.
(b) Discuss how $c'$ would be chosen in part (a) to ensure the test is level $\alpha$. A brief discussion will suffice.

6. Suppose that $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 > 0$ is unknown.
(a) Derive the uniformly most powerful (UMP) level $\alpha$ test for testing

$$H_0 : \sigma^2 \geq \sigma_0^2$$
$$\text{versus}$$
$$H_1 : \sigma^2 < \sigma_0^2.$$

(b) Derive an expression for the power function $\beta(\sigma^2)$ of the UMP test.