From Casella and Berger, do the following problems from Chapter 7:

Homework 4: 1, 4, 6, 10, and 11.
Homework 5: 19, 23, 26, 38, and 40.
Homework 6: 46-50 and 52.

These are extra problems that I have given on past exams (in STAT 713 or in related courses). You do not have to turn these in.

7.1. Suppose that $X_1, ..., X_n, Y_1, ..., Y_n$ are mutually independent random variables. The X_i's have an exponential distribution with mean σ/θ_i and the Y_i's have an exponential distribution with mean $\sigma\theta_i$. The $n+1$ parameters $\theta_1, ..., \theta_n$ and σ are all unknown.
(a) Write down the likelihood function $L(\theta_1, \ldots, \theta_n, \sigma)$.
(b) Show that the MLE of σ is

$$\widehat{\sigma}_n = n^{-1} \sum_{i=1}^{n} (X_i Y_i)^{1/2}.$$

(c) Show that $\widehat{\sigma}_n \xrightarrow{p} (\pi/4)\sigma$, as $n \to \infty$.

7.2. Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from $f(y|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}^p$. Let $\widehat{\theta}$ denote the maximum likelihood estimator of θ and let $\widehat{\theta}_{(i)}$ denote the maximum likelihood estimator of θ when the ith observation Y_i is deleted from the sample. To test model misspecification using the data $Y_1, Y_2, ..., Y_n$, Presnell and Boos (*Journal of the American Statistical Association*, **99**, 216-227) propose the logarithm of the "in and out of sample" (IOS) likelihood ratio, given by

$$\text{IOS} = \log \left\{ \frac{\prod_{i=1}^{n} f(Y_i|\widehat{\theta})}{\prod_{i=1}^{n} f(Y_i|\widehat{\theta}_{(i)})} \right\}.$$

(a) Show that the IOS statistic can be rewritten as

$$\text{IOS} = \sum_{i=1}^{n} \{ l(Y_i; \widehat{\theta}) - l(Y_i; \widehat{\theta}_{(i)}) \},$$

where $l(y; \theta) = \log f(y|\theta)$.
(b) Take $p = 1$ and suppose that $Y_1, Y_2, ..., Y_n$ are iid Poisson with mean $\theta > 0$. Show that

$$\text{IOS} = \sum_{i=1}^{n} (\overline{Y}_{(i)} - \overline{Y}) + \sum_{i=1}^{n} Y_i \log(\overline{Y}/\overline{Y}_{(i)}) \approx \frac{S^2}{\overline{Y}},$$

where S^2 is the usual sample variance. *Hint:* To show the approximate equality, do the following. First, show that

$$\overline{Y}_{(i)} - \overline{Y} = (\overline{Y} - Y_i)/(n-1).$$

Second, use the first-order Taylor series expansion

$$\log(\overline{Y}_{(i)}) \approx \log(\overline{Y}) + (\overline{Y}_{(i)} - \overline{Y})/\overline{Y}.$$

(c) In part (b), argue that $\text{IOS} \xrightarrow{p} 1$, as $n \to \infty$.

7.3. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from a $\mathcal{U}(0, \theta)$ distribution, where $\theta > 0$. In turn, the parameter θ is best regarded as a random variable with a Pareto(a, b) distribution, that is,

$$\pi(\theta) = \begin{cases} \dfrac{ba^b}{\theta^{b+1}}, & \theta > a \\ 0, & \text{otherwise,} \end{cases}$$

where $a > 0$ and $b > 0$ are known.
(a) Turn the "Bayesian crank" to find the posterior distribution of θ. I would probably start by working with a sufficient statistic.
(b) Find the posterior mean and use this as a point estimator of θ.
(c) Can $\widehat{\theta}_B$ be written as a linear combination of the prior mean and the MLE of θ? If so, prove it. If not, show that this can not be done.

7.4. Suppose that $\mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x}|\theta)$, where $\theta \in \Theta$. Suppose that $T = T(\mathbf{X})$ is a sufficient statistic for θ. Prove the following statement: If $W = W(\mathbf{X})$ is the uniformly minimum variance unbiased estimator (UMVUE) of θ, then $W = E(W|T)$ with probability one.

7.5. Suppose $X_1, X_2, ..., X_n$ is an iid sample from $f_X(x|\theta) = \theta e^{-\theta x} I(x > 0)$, where $\theta > 0$.
(a) For $n \geq 2$, show that

$$\widehat{\theta} = \frac{n-1}{n\overline{X}}$$

is the uniformly minimum variance unbiased estimator (UMVUE) of θ.
(b) Calculate $\text{var}_\theta(\widehat{\theta})$. Comment, in particular, on the $n = 2$ case.
(c) Show that $\text{var}_\theta(\widehat{\theta})$ does not attain the Cramer-Rao Lower Bound (CRLB) on the variance of all unbiased estimators of $\tau(\theta) = \theta$.
(d) For this part only, suppose that $n = 1$. If $T(X_1)$ is an unbiased estimator of θ, show that $P_\theta(T(X_1) < 0) > 0$.

7.6. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\sigma) = \frac{1}{2\sigma} e^{-|x|/\sigma} I(-\infty < x < \infty),$$

where $\theta > 0$.
(a) Show that

$$T(\mathbf{X}) = \sum_{i=1}^{n} |X_i|$$

is a complete and sufficient statistic for σ.
(b) Prove that $Y_1 = |X_1|$ follows an exponential distribution with mean σ.
(c) Find the uniformly minimum variance unbiased estimator (UMVUE) of $\tau(\sigma) = \sigma^r$, where r is a fixed constant larger than 0.

7.7. Suppose $X_1, X_2, ..., X_n$ is an iid sample from $f_X(x|\theta) = \theta(1-x)^{\theta-1} I(0 < x < 1)$, where $\theta > 0$.
(a) Find the method of moments (MOM) estimator of θ.

(b) Find the maximum likelihood estimator (MLE) of θ.
(c) Find the MLE of $P_\theta(X > 1/2)$.
(d) Is there a function of θ, say $\tau(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao Lower Bound? If so, find it and identify the corresponding estimator. If not, show why not.

7.8. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \begin{cases} \dfrac{-(1-\theta)^x}{x \ln \theta}, & x = 1, 2, 3..., \\ 0, & \text{otherwise}, \end{cases}$$

where the parameter θ satisfies $0 < \theta < 1$.
(a) Estimate θ using the method of moments (MOM) and using the method of maximum likelihood. **Note:** I am not sure if you can get closed form expressions for either estimator, but that is OK. Just write out the equation(s) that would need to be solved (numerically) to find the estimator. This is typically how estimators are calculated in "real" research problems (numerically).
(b) I used R to sample $n = 20$ observations from this distribution. Here are the outcomes:

```
> sample(x,20,replace=T,prob=prob)
 [1] 1 2 8 1 1 2 2 4 1 7 1 1 1 1 2 4 3 1 6 1
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Using your work in part (a), calculate both the MOM and MLE based on these data. Your answers should be numerical.

7.9. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from the probability density function (pdf) given by

$$f_X(x|\beta) = \frac{1}{\Gamma(m)\beta^m} x^{m-1} e^{-x/\beta} I(x > 0),$$

where $\beta > 0$ is unknown and m is a known constant larger than 1.
(a) Show that $T = T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a complete and sufficient statistic for $\{f_X(x|\beta) : \beta > 0\}$.
(b) Show that

$$E_\beta\left(\frac{m-1}{X_1}\right) = \frac{1}{\beta}.$$

(c) For $t > 0$, show that the conditional density of X_1, given $T = t$, is

$$f_{X_1|T}(x_1|t) = \frac{\Gamma(mn)}{\Gamma(m)\Gamma\{m(n-1)\}} \frac{1}{t} \left(\frac{x_1}{t}\right)^{m-1} \left(1 - \frac{x_1}{t}\right)^{m(n-1)-1} I(0 < x_1 < t).$$

(d) Show that

$$E\left(\frac{m-1}{X_1}\middle| T = t\right) = \frac{mn-1}{t}.$$

(e) What is the UMVUE for $\tau(\beta) = 1/\beta$?

7.10. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from $f_X(x|\theta)$, where $\theta \in \Theta$. In each case below, find (i) the method of moments estimator of θ, (ii) the maximum likelihood estimator of θ, and (iii) the uniformly minimum variance unbiased estimator (UMVUE) of $\tau(\theta)$.

(a)

$$f_X(x|\theta) = \frac{1}{\sqrt{2\pi}\theta x} \exp\left[-\frac{(\log x)^2}{2\theta^2}\right] I(x > 0), \qquad \Theta = \{\theta : \theta > 0\}, \qquad \tau(\theta) = \theta^2.$$

(b)

$$f_X(x|\theta) = \binom{m}{x} \theta^x (1-\theta)^{m-x} I(x = 0, 1, ..., m), \qquad \Theta = \{\theta : 0 < \theta < 1\}, \qquad \tau(\theta) = P_\theta(X = m).$$

(c)

$$f_X(x|\theta) = \frac{1}{2\theta} I(0 < x < 2\theta), \qquad \Theta = \{\theta : \theta > 0\}, \qquad \tau(\theta) \text{ arbitrary, differentiable}$$

(d) $n = 1$ (sample size of $n = 1$ only)

$$f_X(x|\theta) = \frac{\theta^x e^{-\theta}}{x!} I(x = 0, 1, 2, ...,), \qquad \Theta = \{\theta : \theta > 0\}, \qquad \tau(\theta) = e^{-2\theta}$$

In part (d), comment on whether the UMVUE for $\tau(\theta) = e^{-2\theta}$ makes sense.

7.11. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from the probability mass function (pmf) given by

$$f_X(x|\theta) = \begin{cases} (1-\theta)\theta^x, & x = 0, 1, 2, ... \\ 0, & \text{otherwise}, \end{cases}$$

where $0 < \theta < 1$.

(a) Find the maximum likelihood estimator of θ.

(b) Find the Cramer-Rao Lower Bound (CRLB) on the variance of unbiased estimators of $E_\theta(X)$. Can this lower bound be attained?

(c) Find the method of moments estimator of θ.

(d) Put a beta$(2, 3)$ prior distribution on θ. Find the posterior mean. Treating this as a frequentist estimator, find its limit (in probability) as $n \to \infty$.

7.12. Suppose that $X_1, X_2, ..., X_n$ are iid Poisson observations, each having common pdf

$$f_X(x|\theta) = \begin{cases} \dfrac{\theta^x e^{-\theta}}{x!}, & x = 0, 1, 2, ... \\ 0, & \text{otherwise}. \end{cases}$$

Find the UMVUE of $\tau(\theta) = \theta^2$.

7.13. Suppose $X_1, X_2, ..., X_n$ is an iid sample from a uniform distribution over $(\theta,\ \theta+|\theta|)$, where $\theta \neq 0$.
(a) Find the method of moments estimator of θ.
(b) Find the maximum likelihood estimator (MLE) of θ.
(c) Is the MLE of θ a consistent estimator of θ? Explain.

7.14. Consider data that follow an exponential regression with no intercept

$$Y_i \overset{ind}{\sim} \exp(\beta x_i),$$

where the scalar parameter $\beta > 0$ is unknown and the x_i's > 0 are fixed and known for $i = 1, \ldots, n$. That is, $Y_1, \ldots, Y_n$ are independent random variables with density functions

$$f_{Y_i}(y) = \frac{1}{\beta x_i} \exp\left(-\frac{y}{\beta x_i}\right),$$

for $y > 0$. Note that $E(Y_i) = \beta x_i$.
(a) Derive the least squares estimator $\widetilde{\beta}$, i.e., minimize

$$Q(\beta) = \sum_{i=1}^{n} (Y_i - \beta x_i)^2.$$

What are the mean and variance of this estimator?
(b) Derive the maximum likelihood estimator $\widehat{\beta}$.
(c) What is the exact (i.e., finite-sample) sampling distribution of $\widehat{\beta}$?
(d) Which estimator, $\widetilde{\beta}$ or $\widehat{\beta}$, has smaller variance and why?

7.15. Suppose that $X_1, X_2, ..., X_n$ is an iid sample, each with probability p of being distributed as uniform over $(-1/2, 1/2)$ and with probability $1 - p$ of being distributed as uniform over $(0, 1)$.
(a) Find the cumulative distribution function (cdf) and the probability density function (pdf) of X_1.
(b) Find the maximum likelihood estimator (MLE) of p.
(c) Find another estimator of p using the method of moments (MOM).

7.16. Suppose that $X_1, X_2, ..., X_n$ is an iid sample of size n from a Pareto pdf of the form

$$f_X(x|\theta) = \begin{cases} \theta x^{-(\theta+1)}, & x > 1 \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta > 0$.
(a) Find $\widetilde{\theta}$, the method of moments (MOM) estimator for θ. For what values of θ does $\widetilde{\theta}$ exist? Why?
(b) Find $\widehat{\theta}$, the maximum likelihood estimator (MLE) for θ.
(c) Show explicitly that the MLE depends on the sufficient statistic for this Pareto family but that the MOM estimator does not.

7.17. Insurance payments data are typically highly positively skewed and distributed with a larger upper tail. A reasonable model for this type of data is the Weibull distribution. Specifically, suppose that $X_1, X_2, ..., X_n$ denote n payments, modeled as iid random variables with common Weibull pdf

$$f_X(x|\theta) = \begin{cases} \frac{m}{\theta} x^{m-1} e^{-x^m/\theta}, & x > 0 \\ 0, & \text{otherwise}, \end{cases}$$

where $m > 0$ is known and θ is unknown. In turn, suppose that $\theta \sim \text{IG}(\alpha, \beta)$, that is, θ has an inverted gamma (prior) pdf

$$\pi(\theta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{\theta^{\alpha+1}} e^{-1/\beta\theta}, & \theta > 0 \\ 0, & \text{otherwise}. \end{cases}$$

(a) Prove that the inverted gamma $\text{IG}(\alpha, \beta)$ prior is a conjugate prior for the Weibull family above.
(b) Suppose that $m = 2$, $\alpha = 0.5$, and $\beta = 2$. Here are $n = 10$ insurance payments (measured in $10,000s):

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0.2697 0.0719 0.4698 0.8192 3.9700 0.2681 0.2415 2.8351 0.0885 0.1180
```

Find the posterior mean of θ.