1. Suppose that $\mathbf{A}$ is an $n \times n$ symmetric matrix. Prove that $\mathbf{A}$ is idempotent if and only if $r(\mathbf{A}) + r(\mathbf{I} - \mathbf{A}) = n$.

2. Let $\mathbf{P}$ and $\mathbf{A}$ be $n \times n$ matrices. Define $\mathbf{D} = \mathbf{P}'\mathbf{A}\mathbf{P}$. Show that if $\mathbf{A}$ is nnd, then so is $\mathbf{D}$.

3. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

(a) Show that $\mathbf{A}$ is pd.
(b) Compute $\mathbf{A}^{1/2}$, the symmetric square root of $\mathbf{A}$. Check your work by showing that $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$.

4. Suppose that $\mathbf{A}_{n \times n}$ is a symmetric with eigenvalues $\lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(n)}$. Prove that

$$\sup_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_{(n)}.$$

5. Prove that if a matrix $\mathbf{A}$ is pd, then $\mathbf{A}^{-1}$ is also pd.

6. Define

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Suppose that $\mathbf{M}$ is the perpendicular projection matrix onto $\mathcal{C}(\mathbf{A})$. Find $r(\mathbf{M})$ and $tr(\mathbf{M})$.

7. Suppose that $\mathbf{Y} = (Y_1, Y_2, Y_3)'$ has mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ given by

$$\boldsymbol{\mu} = \begin{pmatrix} 4 \\ 6 \\ 10 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 8 & 5 & 0 \\ 5 & 12 & 4 \\ 0 & 4 & 9 \end{pmatrix}.$$

(a) Find the mean and variance of $Z = Y_1 - Y_2 + Y_3$.
(b) Let

$$\mathbf{A} = \begin{pmatrix} 3 & 5 & 4 \\ 1 & 2 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Find $E(\mathbf{AY} + \mathbf{b})$ and $\text{cov}(\mathbf{AY} + \mathbf{b})$.

8. Suppose that $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)'$ is a random vector with covariance matrix $\boldsymbol{\Sigma} = \text{cov}(\mathbf{Y})$, and let $\mathbf{a}$ and $\mathbf{b}$ be conformable vectors of constants. Prove that

$$\text{cov}(\mathbf{a}'\mathbf{Y}, \mathbf{b}'\mathbf{Y}) = \mathbf{a}'\boldsymbol{\Sigma}\mathbf{b}.$$

9. Suppose that $\mathbf{Y}_{n\times 1}$ and $\mathbf{X}_{k\times 1}$ are random vectors. Define $\mathbf{Z} = \mathbf{Y} - E(\mathbf{Y}|\mathbf{X})$. Show that $\mathbf{Z}$ and $\mathbf{X}$ are uncorrelated.

10. Suppose that $\mathbf{Y}$ and $\mathbf{X}$ are random vectors with means $\boldsymbol{\mu}_{\mathbf{Y}}$ and $\boldsymbol{\mu}_{\mathbf{X}}$, respectively, variance matrices $\boldsymbol{\Sigma}_{\mathbf{Y}}$ and $\boldsymbol{\Sigma}_{\mathbf{X}}$, respectively, and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{YX}}$. Assume that $\boldsymbol{\Sigma}_{\mathbf{X}}$ is nonsingular. Define

$$\mathbf{W} = \boldsymbol{\mu}_{\mathbf{Y}} + \boldsymbol{\Sigma}_{\mathbf{YX}}\boldsymbol{\Sigma}_{\mathbf{X}}^{-1}(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})$$

and $\mathbf{Z} = \mathbf{Y} - \mathbf{W}$. Derive $\text{cov}(\mathbf{Z})$ and show that $\text{cov}(\mathbf{Z}) \leq_{\text{pd}} \boldsymbol{\Sigma}_{\mathbf{Y}}$, with equality when $\boldsymbol{\Sigma}_{\mathbf{YX}} = \mathbf{0}$.

11. Consider the mixed-effects linear model

$$\mathbf{Y}_{n\times 1} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}_1\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2,$$

where $\mathbf{X}$ is $n \times p$, $\boldsymbol{\beta}$ is $p \times 1$, $\boldsymbol{\epsilon}_1$ has mean vector $\mathbf{0}_{r\times 1}$ and variance-covariance matrix $\boldsymbol{\Sigma}_1$, and $\boldsymbol{\epsilon}_2$ has mean vector $\mathbf{0}_{n\times 1}$ and variance-covariance matrix $\sigma^2\mathbf{I}_n$. Also, assume that $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are uncorrelated.
(a) Compute $\text{cov}(\mathbf{Y})$.
(b) ($\uparrow$) Specialize to the **one-factor random-effects model**

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

for $i = 1, 2, 3$ and $j = 1, 2$, where $\alpha_1, \alpha_2, \alpha_3$ are iid $\mathcal{N}(0, \sigma_\alpha^2)$, ϵ_{ij} are iid $\mathcal{N}(0, \sigma^2)$, and the α_i's and ϵ_{ij}'s are mutually independent. Put this model into the form $\mathbf{Y}_{n\times 1} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}_1\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2$, and compute $\text{cov}(\mathbf{Y})$.