STAT 515 – Sections 5.6 Supplement

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Last Updated: September 9, 2016

S5.6 – Relating the Poisson and Exponential

Section 4.5 introduces the Poisson random variable with 4 basic characteristics. We say that something that follows these rules follows a Poisson process.

1) The experiment consists of counting the number of times (let's use Y this time) that a certain event occurs during some given amount of time (or area, volume, weight, distance, or any other unit of measurement).

2) The probability that an event occurs in any unit of measurement is the same for all units of that size.

3) The number of events that occur in one unit of measurement is independent of the number that occur in other units.

4) The mean number of events that occur in one unit of measurement is denoted by the Greek letter lambda (λ).

As given on page 215 the probability mass function for a Poisson random variable is $P[Y = y] = \frac{\lambda^y e^{-\lambda}}{y!}$ for x=0, 1, 2, …. .

Imagine that we have something that follows a Poisson process with parameter λ/unit, but we want to know about the random variable X=amount of time until the first event. One way to approach this is to try and figure out the cumulative distribution function for X.

$P[X \leq x]$ = P[1st failure happens before the x^{th} unit of time]

= 1 – P[1st failure happens after the x^{th} unit of time]

= 1 – P[there are 0 failures before time x]

The probability of having no failures before a given time is a Poisson random variable! The only complication is adjusting the rate to be over x units of time instead of 1 unit.

$$\frac{\lambda}{(1\ unit\ of\ time)} = \frac{x}{x} \cdot \frac{\lambda}{(1\ unit\ of\ time)} = \frac{x\lambda}{(x\ units\ of\ time)}$$

So this is the probability of seeing 0 failures for a Poisson distribution with parameter $x\lambda$.

$P[X \leq x] = 1 -$ P[there are 0 failures before time x]

$$= 1 - \frac{(x\lambda)^0 e^{-x\lambda}}{0!}$$

$$= 1 - e^{-x\lambda}$$

This also gives

$$P[X \geq x] = P[X > x] = 1 - P[X \leq x] = e^{-x\lambda}$$

Compare this to the formula for Figure 5.28 in the middle of page 268. If we set $\lambda=1/\theta$ this is exactly the same as an exponential distribution!

So there are two ways to write out (or "parameterize") an exponential distribution:

Parameterization 1: If λ is the parameter for a Poisson process (the number of events expected in each unit of time), then:

$$f(y) = \lambda e^{-x\lambda} \qquad P[X \leq x] = 1 - e^{-x\lambda} \qquad \mu = \frac{1}{\lambda} \qquad \sigma = 1/\lambda$$

This parameterization with lambda is the one that R uses (lambda is the "rate parameter").

Parameterization 2: If θ is the expected amount of time until the first occurrence, then:

$$f(y) = \frac{1}{\theta} e^{-x/\theta} \qquad P[X \leq x] = 1 - e^{-x/\theta} \qquad \mu = \theta \qquad \sigma = \theta$$

In R, you need to use $1/\theta$ as the rate parameter. To answer example 5.14 in R we would use 1-P[X ≤ 5] = 1-`pexp(5,1/2)` where the ½ is 1/θ.

The particular parameterization you should use depends on what information you were given. Were you told how many events were expected to occur in a unit of time (λ) or how long you expected to wait until the first event (θ)?

You can also use the relationship $\lambda=1/\theta$ to switch between the two ways of looking at it.

Example 5.15 (page 269) continued
e) What is the λ parameter of the Poisson process that the magnetron tubes life-spans follow?
f) What is the expected number of tube failures in one year?
g) Using R, what is the probability of observing two or fewer failures in one year?

Solution: θ = 6.25 years is the average lifespan, so the parameter $\lambda=1/\theta$=1/6.25years = 0.16/year since λ is the mean of the Poisson distribution. The expected number of failures in one year is 0.16. Using R to find P[Y≤ 2] for a Poisson 0.16, would be `ppois(2,.16)` which is approximately 99.94%

Example 4.14 (page 215) continued:
e) What is the expected amount of time until the first whale is observed?
f) What is the probability that the first whale will be observed by half-way through the week?

Solution: This is a Poisson process with λ=2.6/week. The expected time until the first whale is observed is the expected value of the exponential distribution with this parameter, so $1/\lambda$ = 1/(2.6/week) = week/0.385 ≈ 0.385 weeks which is around 2.7 days. The probability that the first whale will be observed by half-way through the week is P[X≤ 0.5] for the exponential distribution = $1 - e^{-\lambda x} = 1 - e^{-2.6(0.5)} = 1 - e^{-1.3} \approx 0.727$. This could also be found using `pexp(0.5,2.6)` in R.