COUNTING RULES

For experiments for which all sample points (outcomes) are equally likely, we compute

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points}\}}$$

<u>Theorem</u> (Fundamental theorem of counting):

If a job consists of K tasks, such that the tasks may be completed in $n_1, \ldots, n_K$ ways, then there are

"times"

$$n_1 * \cdots * n_K$$

ways to do the job.

Many counting situations are like drawing r things from N things...

"order matters to us"

I. ordered, without replacement (OwoR)
II. unordered, without replacement (UwoR)
III. ordered, with replacement (OwR)
IV. unordered, with replacement (UwR)

"put back what you draw each time"

"order does not matter to us"

<u>Eg.</u> 15 dollars to be disbursed among students...

I. Top 5 of 60 students to recieve \$5, \$4, \$3, \$2, \$1 according to rank

(OwoR)

$$\# \text{ ways} = 60 * 59 * 58 * 57 * 56 = \frac{60!}{55!} = 655{,}381{,}440$$

<u>Formula</u>: Number of ways to draw r things from N, ordered, without replacement, is

$$\frac{N!}{(N-r)!}$$

II. Top 5 of 60 students each to recieve \$3.

(UWoR)

$$\# \text{ ways} = \frac{60!}{55!} \Big/ 5!$$

Divide by # redundant orderings (permutations) (There are 5! ways in which the top 5 students can be ranked.)

$$= 5{,}461{,}512$$

Formula: Number of ways to draw r things from N, unordered, without replacement, is

$$\underbrace{\binom{N}{r}}_{\text{"N choose r"}} := \frac{N!}{(N-r)!\,r!}$$

we use ":=" to define notation

III. Student chosen at random during each of 5 class periods.

(OWR) Chosen student recieves according to the day the amounts

Day 1	Day 2	Day 3	Day 4	Day 5
\$6.82	\$4.36	\$2.45	\$1.09	\$.28

\# ways to choose 5 students from 60, with replacement when the order matters is

$$60 * 60 * 60 * 60 * 60 = 60^5 = 777{,}600{,}000$$

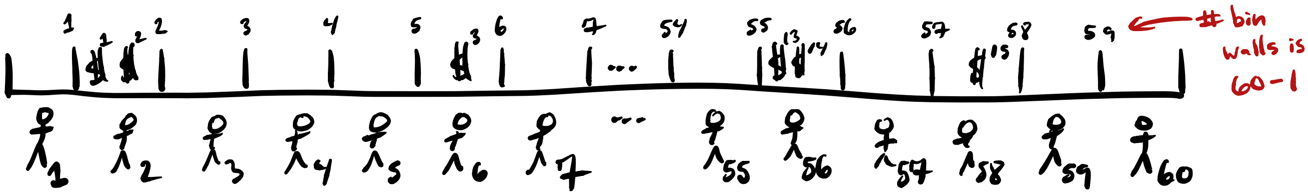
Formula: Number of ways to draw r things from N, ordered with replacement, is

$$N^r$$

IV. 15 dollar bills distributed among 60 students.

(UWR) Like dropping 15 dollar bills into 60 bins;

Each unique possibility corresponds to an arrangement of the bin walls and the dollar bills:



We can arrange the bin walls and dollar bills in this many ways:

$$\frac{(\overbrace{60-1}^{\#\text{ bin walls}} + \overbrace{15}^{\#\text{ dollar bills}})!}{\underbrace{(60-1)!}_{\#\text{ orderings of bin walls}}\ \underbrace{15!}_{\#\text{ orderings of bills}}} \approx 1.824 \times 10^{15}$$

Formula: Number of ways to draw r things from N, unordered, with replacement, is

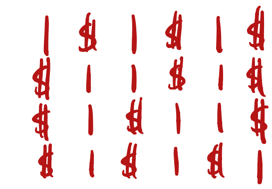
$$\frac{(N-1+r)!}{(N-1)!\,r!} = \binom{N+r-1}{r}$$

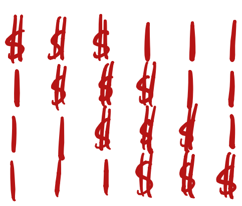
Smaller example: Distribute 3 dollar bills among 4 people

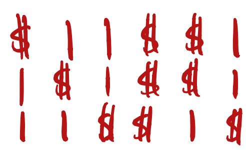
Enumerate all possibilities:

$$\underbrace{\left(\begin{matrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{matrix}\right.}_{1\text{ person gets }\$3}
\qquad
\underbrace{\begin{matrix}2&1&0&0\\2&0&1&0\\2&0&0&1\\ \\1&2&0&0\\0&2&1&0\\0&2&0&1\\ \\1&0&2&0\\0&1&2&0\\0&0&2&1\\ \\1&0&0&2\\0&1&0&2\\0&0&1&2\end{matrix}}_{1\text{ person gets }\$2}
\qquad
\underbrace{\left.\begin{matrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{matrix}\right)}_{1\text{ person gets }\$0}$$

Each possibility corresponds to a permutation of $$$ and bin walls:







There are 20 possibilities, which agrees with the formula:

$$\frac{(4-1+3)!}{(4-1)!\,3!} = \frac{6!}{3!3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\quad 3\cdot2\cdot1} = 5\cdot4 = 20$$

In summary, the numbers of ways to draw r things from N are:

	without replacement	with replacement
Ordered	$\frac{N!}{(N-r)!}$	N^r
Unordered	$\binom{N}{r}$	$\binom{N+r-1}{r}$

Some poker examples: 5 cards drawn from 52-card deck

1. # hands with ace of diamonds

   - # of ways to draw 4 cards from remaining 51 is

$$\binom{51}{4} = \frac{51 \cdot 50 \cdot 49 \cdot 48}{4 \cdot 3 \cdot 2} = 17 \cdot 25 \cdot 49 \cdot 12 = 249{,}900$$

2. # full house hands

   - draw 5 cards from 52
   - full house: 3 cards of one rank and 2 of another

There are $13 * 12$ ordered pairs of ranks

Each pair of ranks can be dealt in

$$\binom{4}{3} * \binom{4}{2} \text{ ways}$$

choose 2 of the 4 eights, for example.

Choose 3 of the 4 Aces, for example

So the total # of full house hands is

$$\binom{4}{3} * \binom{4}{2} * 13 * 12 = 3{,}744$$

3. # flush hands

- flush: 5 cards of same suit but not sequential

4 possible suits.

For each suit, $\underbrace{\binom{13}{5}}_{\text{\# ways to draw 5 cards from 13}} - 10$ flushes possible

\# of sequences:
1-5
⋮
10-14
} 10 sequences (Ace can be high or low)

So # flush hands is

$$4 * \left[\binom{13}{5} - 10\right] = 5{,}108$$

4. # straight hands

- Straight: 5 cards in sequence, not all of same suit

There are 10 sequences possible, 1-5, ..., 10-14

Can make 1-5 with

(4 choices of Ace) × (4 choices of two) × ... × (4 choices of five)

So # ways to make 1-5 is 4^5.

But 4 of these are sequences of a single suit.

So # straight hands is

$$\underbrace{10}_{\text{\# possible sequences}} * \underbrace{(4^5 - 4)}_{\substack{\text{for each sequence,} \\ \text{\# ways minus} \\ \text{\# ways with same suit.}}} = 10{,}200.$$

Application of counting to probability

Recall the point of counting:

For experiments for which all sample points (outcomes) are equally likely, we compute

$$P(A) = \frac{\#\{\text{sample points in } A\}}{\#\{\text{sample points}\}}$$

Probability for poker examples:

Experiment: Draw 5 cards from 52-card deck

Sample space

$$S = \left\{ \begin{array}{c} \text{All possible} \\ \text{poker hands} \end{array} \right\}$$

How many sample points in S?

$$\binom{52}{5} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 26\cdot 17\cdot 10\cdot 49\cdot 12 = 2{,}598{,}960$$

possible hands

1. $P(\text{get Ace of diamonds}) = \dfrac{249{,}900}{2{,}598{,}960} = .0962$ (249,900: #hands w/ Ace of diamonds)

2. $P(\text{full house}) = 3{,}744/2{,}598{,}960 = .001441$

3. $P(\text{flush}) = 5{,}108/2{,}598{,}960 = .001965$

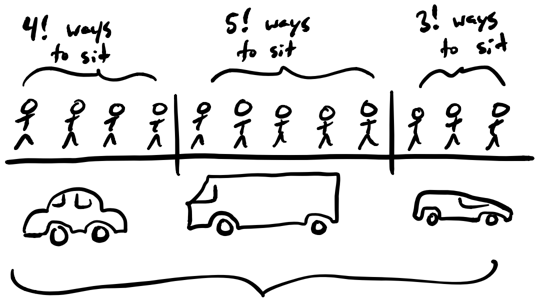
4. $P(\text{straight}) = 10{,}200/2{,}598{,}960 = .003925$

One more counting rule (Partitioning)

Formula: The number of ways to partition N things into K groups of sizes $n_1, \ldots, n_K$, where $n_1 + \cdots + n_K = N$, is

$$\frac{N!}{n_1! \cdot \ldots \cdot n_K!}$$

Eg: 12 passengers traveling in 3 vehicles taking 4, 5, and 3 passengers, resp.



12! ways to sit in all 12 passenger seats.

So # ways to split up (partition) is

$$\frac{12!}{4!\,5!\,3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 3 \cdot 11 \cdot 5 \cdot 3 \cdot 8 \cdot 7 = 27{,}720$$

Eg: (prev. cont.) If every partition equally likely, then

$$P(\text{You ride with friend}) = \frac{\#\{\text{ways you ride with friend}\}}{27{,}720}$$

$\#\{\text{You two in } 1^{st} \text{ car}\}$:

\# seating arrangements for remaining people

$$\frac{10!}{2!\,5!\,3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 5 \cdot 9 \cdot 8 \cdot 7 = 2{,}520$$

\# remaining sit arrangement in your car

$\#\{\text{You two in } 2^{nd} \text{ car}\}$: $$\frac{10!}{4!\,3!\,3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 5 \cdot 3 \cdot 8 \cdot 7 \cdot 5 = 4{,}200$$

$\#\{\text{You two in } 3^{rd} \text{ car}\}$: $\dfrac{10!}{4!\,5!\,2!} = \dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 5\cdot 3\cdot 2\cdot 7\cdot 6 = 1{,}260$

So $\#\{\text{ways you ride w/ friend}\} = 2520 + 4200 + 1260 = 7{,}980$

and

$$P(\text{You ride with friend}) = \frac{7{,}980}{27{,}720} = .2879$$