STAT 511 fa 2019 Lec 03 slides

Conditional probability and independence

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

- So far we have considered unconditional probabilities.
- We have computed $P(A)$ with respect to a sample space $\mathcal{S}$ with $P(\mathcal{S}) = 1$
- We can sometimes update $\mathcal{S}$ based on new information...

Exercise: Consider the experiment of rolling two dice, with sample space:

$$\mathcal{S} = \left\{ \begin{array}{llllll} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6), \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6), \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

Let

$$\begin{aligned} A &= \text{roll doubles} \\ B &= \text{absolute difference in rolls less than 2} \\ C &= \text{sum of rolls 10 or more} \end{aligned}$$

1. Find $P(A)$.
2. Suppose we know that B has occurred. *Then* what is the probability of A?
3. Suppose we know that C has occurred. *Then* what is the probability of A?

Conditional probability

For $A, B \subset \mathcal{S}$ with $P(B) > 0$, the *conditional probability* of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If we know that B happens, then B becomes the updated sample space.

Exercise: For example on previous slide, use above to find

1. $P(A|B)$
2. $P(A|C)$
3. $P(B|A)$
4. $P(C|A)$
5. $P(B|C)$

Exercise: Consider the the events

$$A = \text{Oversleep on day 1 of semester}$$
$$B = \text{Oversleep on day 2 of semester}$$

and suppose $P(A) = 1/4$, $P(B) = 1/3$, and $P(A \cap B) = 1/10$.

Find

1. $P(A|B)$
2. $P(B|A)$
3. $P(B|A^c)$

Bayes' Rule

Let $A_1, A_2, \ldots$ be a partition of the sample space and let B be any set with $P(B) > 0$. Then

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)P(A_j)}.$$

This is called *Bayes' rule*.

Draw picture to illustrate.

Application (simpler version) of Bayes' Rule

If A and B are events in $\mathcal{S}$ and $P(B) > 0$, then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

An application of Bayes' rule because A and A^c form a partition of S.

Exercise: Suppose there is noisy Morse code transmission such that

$$P(\text{dash received}|\text{dot sent}) = 1/8$$
$$P(\text{dot received}|\text{dash sent}) = 1/8$$

In addition, suppose dots and dashes tend to be sent in the ratio 3:4.

Find

1. $P(\text{dot sent}|\text{dot received})$
2. $P(\text{dash sent}|\text{dash received})$

Exercise: Consider an imperfect test for the presence of an infection such that

$$P(\text{test positive}|\text{infection present}) = 0.92 \quad (\text{Sensitivity})$$
$$P(\text{test negative}|\text{infection absent}) = 0.98 \quad (\text{Specificity})$$

Suppose the infection is present in 5% of the population.

If an individual randomly selected from the population is tested, find

1. $P(\text{infection present}|\text{test positive})$
2. $P(\text{infection absent}|\text{test negative})$

Independence

Two events A and B are called *independent* if

$$P(A \cap B) = P(A)P(B).$$

Equivalent definitions of independence

The following statements are equivalent:

- $P(A \cap B) = P(A)P(B)$
- $P(A) = P(A|B)$
- $P(B) = P(B|A)$

Also: If A, B independent, so are the pairs of events A, B^c and A^c, B^c and A^c, B^c.

Exercise: Flip a coin twice and let

$$H_1 = \text{heads on first flip}$$
$$H_2 = \text{heads on second flip}$$

Find $P(H_1 \cap H_2)$ assuming that the flips are independent.

Mutual independence

A collection of events $A_1, A_2, \ldots$ are *mutually independent* if for any subcollection $A_{i_1}, \ldots, A_{i_K}$, we have

$$P\left(\bigcap_{j=1}^{K} A_{i_j}\right) = \prod_{j=1}^{K} P(A_{i_j})$$

Extends notion of independence between two events to independence among a collection of events.

Exercise: Flip a coin n times and let

$$H_i = \text{heads on flip } i, \quad i = 1, \ldots, n.$$

Find $P(\text{all heads})$ assuming that the flips are mutually independent.

Exercise: Use the following for reference:

$$\begin{aligned} p_{O-} &= 0.066, & p_{B-} &= 0.015 \\ p_{O+} &= 0.374, & p_{B+} &= 0.085 \\ p_{A-} &= 0.063, & p_{AB-} &= 0.006 \\ p_{A+} &= 0.357, & p_{AB+} &= 0.034 \end{aligned}$$

		Donor							
		O−	O+	A−	A+	B−	B+	AB−	AB+
Recipient	O−	✓	.	.	.	.	.	.	.
	O+	✓	✓	.	.	.	.	.	.
	A−	✓	.	✓	.	.	.	.	.
	A+	✓	✓	✓	✓	.	.	.	.
	B−	✓	.	.	.	✓	.	.	.
	B+	✓	✓	.	.	✓	✓	.	.
	AB−	✓	.	✓	.	✓	.	✓	.
	AB+	✓	✓	✓	✓	✓	✓	✓	✓

Find the probabilities of the following events:

1. Two randomly selected people can donate and receive between each other.
2. Four randomly selected people are all Rh+.
3. Two out of four randomly selected people have the A antigen in the RBCs.