PROBABILITY MASS AND DENSITY FUNCTIONS

X discrete (i.e. F_X a step function)

Defn: The probability mass function (pmf) p_X of a discrete r.v. X with probability distribution P_X is defined as
$$p_X(x) = P_X(X=x) \quad \text{for all } x \in \mathbb{R}$$

* Values of p_X are the sizes of the jumps in F_X
* We get $p_X(x) = 0$ if $x \notin \mathcal{X}$, since

$$p_X(x) = P_X(X=x) = P\big(\underbrace{\{s \in S : X(s) = x\}}_{=\emptyset \text{ if } x \notin \mathcal{X}}\big) = 0 \text{ for } x \notin \mathcal{X}$$

$= \emptyset$ if $x \notin \mathcal{X}$, and $P(\emptyset) = 0$

E.g. (Bernoulli distribution) One trial with outcomes "success" and "failure". Probability of success equal to p.

$S = \{\text{success}, \text{failure}\}$

Assign $p \in [0,1]$ to success, $1-p$ to failure, giving
$$P(\{\text{success}\}) = p$$
$$P(\{\text{failure}\}) = 1-p$$

Define r.v.
$$X := X(s) = \begin{cases} 1 & \text{if } s = \text{success} \\ 0 & \text{if } s = \text{failure} \end{cases}$$

Then the pmf of X is
$$p_X(x) = P_X(X=x) = \begin{cases} P(\{\text{success}\}) & \text{if } x = 1 \\ P(\{\text{failure}\}) & \text{if } x = 0 \\ P(\emptyset) & \text{if } x \notin \{0,1\} \end{cases}$$

$$= \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{if } x \notin \{0,1\} \end{cases}$$

We can write $p_X(x)$ more concisely as

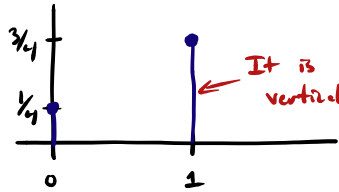
$$p_X(x) = \begin{cases} p^x (1-p)^{1-x} & \text{if } x \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

We tend to ignore the case $x \notin \mathcal{X}$ when writing the pmf, so we just write

$$p_X(x) = p^x (1-p)^{1-x} \quad \text{for } x = 0, 1,$$

and it is understood that $p_X(x) = 0$ for $x \notin \{0,1\}$.

Plot of Bernoulli pmf when $p = 3/4$:



It is convention to draw these vertical bars

E.g. (Geometric distribution) X = # Bernoulli trials w/ independent outcomes to get a success w/ success prob. p

Recall:

x	0	1	2	...
$P_X(X=x)$	p	$\underbrace{(1-p)}_{\text{1 failure}}p$	$\underbrace{(1-p)^2}_{\text{2 failures}} p$	...
	head on first try			

So

$$p_X(x) = (1-p)^{x-1} p \quad \text{for } x = 0, 1, 2, \ldots$$

E.g. (Binomial distribution) X = # successes in n independent Bernoulli trials with success prob. p.

$$S = \{\text{All sequences of successes and failures of length } n\}$$

$$\mathcal{X} = \{0, 1, \ldots, n\}$$

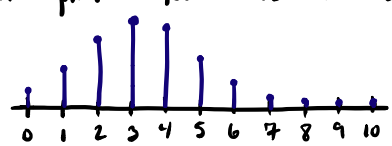
To each sequence with x successes, assign the probability

$$p^x (1-p)^{n-x}$$

There are $\binom{n}{x}$ sequences in S with x successes, so

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n$$

Plot of Binomial pmf for $n = 10$ and $p = 1/3$



X continuous (i.e. F_X a continuous function)

Defn: The probability density function (pdf) f_X of a continuous r.v. X having cdf F_X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \qquad \text{for all } x.$$

* Compare to discrete case: cdf F_X of discrete r.v. X with pmf p_X is

We have summation instead of integration

$$F_X(x) = \sum_{\{t \in \mathcal{X} : t \le x\}} p_X(t)$$

E.g. (Binomial cdf): X = # successes from n indep. Bernoulli trials w/ success prob. p.

$$F_X(x) = \begin{cases} \sum_{t=0}^{\lfloor x \rfloor} \binom{n}{t} p^t (1-p)^{n-t}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

* If f_X satisfying $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ is continuous, then $f_X(x) = \frac{d}{dx} F_X(x)$.

E.g. (Logistic distribution) Let X have cdf

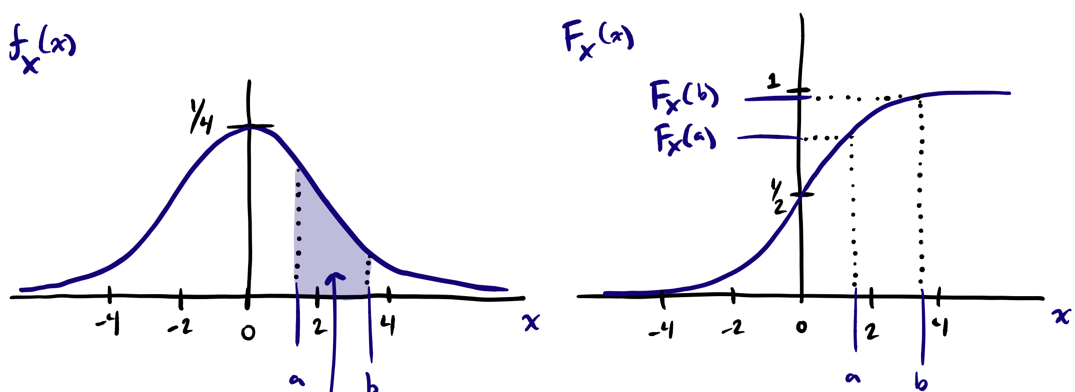
$$F_X(x) = \frac{1}{1+e^{-x}} \quad \text{for all } x$$

Then $\frac{d}{dx} F_X(x) = -\frac{1}{(1+e^{-x})^2} \frac{d}{dx} 1+e^{-x} = -\frac{1}{(1+e^{-x})^2} (-1)\, e^{-x} = \frac{e^{-x}}{(1+e^{-x})^2}$

This is continuous, so the pdf of X is

$$f_X(x) = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{for all } x$$

Illustration: For any $a, b \in \mathbb{R}$



$$P_X(a < X < b) = F_X(b) - F_X(a) = \frac{1}{1+e^{-b}} - \frac{1}{1+e^{-a}}$$

* For any r.v. X with pdf f_X, we have for any $a, b \in \mathbb{R}$

$$P_X(a < X < b) = F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx = \int_{a}^{b} f_X(x)\,dx$$

= Area under pdf between a and b

Theorem: The function p_X is a pmf (f_X is a pdf) iff

(i) $p_X(x) \geq 0$ $\left(f_X(x) \geq 0\right)$ for all x

(ii) $\sum_{x \in \mathcal{X}} p_X(x) = 1$ $\left(\int_{-\infty}^{\infty} f_X(x)\,dx = 1\right)$

E.g. (Poisson Distribution) Consider for some $\lambda > 0$ the function

$$p_X(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & x = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

* Posited as pmf when r.v. X is # occurrences per unit time/space

Show that p_X is a legitimate pmf:

(i) $e^{-\lambda}\lambda^x / x! \geq 0$ for all $x = 0,1,2,\ldots$ ✓

(ii) $$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}_{} = e^{-\lambda} e^{\lambda} = 1$$ ✓

Taylor series expansion of infinitely differentiable function g around x_0:

$$g(x) = \sum_{i=0}^{\infty} \frac{g^{(i)}(x_0)}{i!} (x - x_0)^i,$$

where $g^{(i)}$ is the i^{th} derivative of g, $i = 0,1,2,\ldots$

If $g(x) = e^x$, then $g^{(i)}(x) = e^x$, $i = 0,1,2,\ldots$

Choosing $x_0 = 0$, we get

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{e^0}{i!}(\lambda - 0)^i = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

E.g. (Exponential Distribution) Consider for some $\lambda > 0$ the function

$$f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

* Posited as pdf when r.v. X is time elapsed / space traversed between occurrences of some event

Show that f_X is a legitimate pdf:

(i) $\frac{1}{\lambda} e^{-x/\lambda} \geq 0$ for all $x \geq 0$, so $f_X(x) \geq 0$ for all x ✓

(ii) $$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,dx$$

$$= 0 + \frac{1}{\lambda}\int_0^{\infty} e^{-\frac{x}{\lambda}}\,dx$$

let $u = \frac{x}{\lambda}$, then $\frac{du}{dx} = \frac{1}{\lambda} \iff dx = \lambda du$

$$= \frac{1}{\lambda}\int_0^{\infty} e^{-u} \lambda\,du$$

$$= \int_0^{\infty} e^{-u}\,du$$

$$= 1$$ ✓ So legit...

Notation: For a r.v. X with probability dist. P_X, we often write

- $X \sim F_X$ if P_X has the cdf F_X
- $X \sim p_X$ if P_X has the pmf p_X
- $X \sim f_X$ if P_X has the pdf f_X

Also: For a continuous r.v. $X \sim f_X$,

the support $\mathcal{X}$ of X is $\{x \in \mathbb{R} : f_X(x) > 0\}$

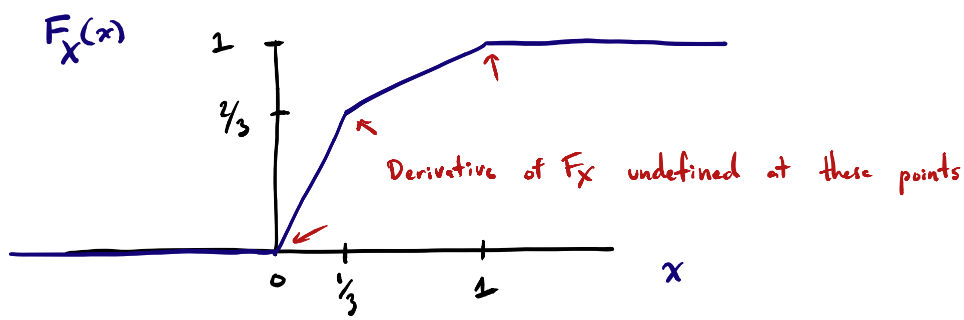
Finding f_X from continuous F_X:

- If F_X has a continuous derivative F_X', then f_X is F_X'
- Otherwise f_X may be piecewise defined:

E.g. Let

$$F_X(x) = \begin{cases} 1 & 1 \le x < \infty \\ \frac{2}{3} + \frac{1}{2}\left(x - \frac{1}{3}\right) & \frac{1}{3} \le x < 1 \\ 2x & 0 \le x < \frac{1}{3} \\ 0 & -\infty < x < 0 \end{cases}$$

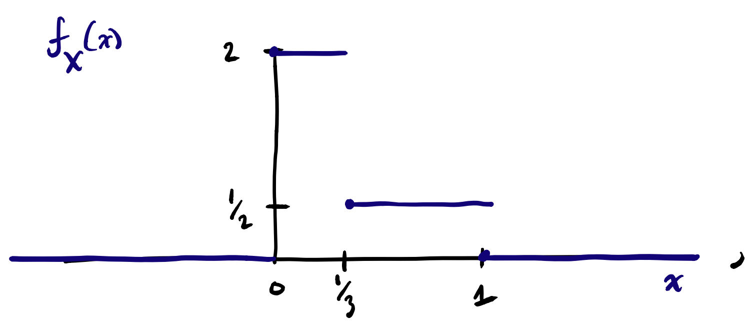
This looks like



We see that with

$$f_X(x) = \begin{cases} 0 & 1 \le x < \infty \\ \frac{1}{2} & \frac{1}{3} \le x < 1 \\ 2 & 0 \le x < \frac{1}{3} \\ 0 & -\infty < x < 0, \end{cases}$$

which looks like



we have $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ for all x.