# STAT 511 fa 2019 Lec 05 slides

## Probability mass and density functions

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

## Probability mass function of a discrete random variable

The *probability mass function (pmf)* $p_X$ of a discrete random variable $X$ with probability distribution $P_X$ is defined as

$$p_X(x) = P_X(X = x) \text{ for all } x \in \mathbb{R}.$$

**Exercise**: Find the pmfs of the following rvs based on independent Bernoulli trials with success probability $p$:

1. $X = 1$ if first trial a success, $X = 0$ if a failure.
2. $X =$ number of successes in $3$ trials.
3. $X =$ number of successes in $n$ trials.
4. $X =$ number of trial on which the first success occurs.

## Probability density function of a continuous random variable

The *probability density function (pdf)* $f_X$ of a continuous rv $X$ with cdf $F_X$ is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \text{ for all } x \in \mathbb{R}.$$

If $f_X$ which satisfies the above is continuous, then $f_X(x) = \frac{d}{dx}F_X(x)$.

Note that the cdf of a discrete rv $X$ with pmf $p_X$ and support $\mathcal{X}$ can be written as

$$F_X(x) = \sum_{\{t\in\mathcal{X}:t\leq x\}} p_X(t).$$

**Exercise:** Let $X$ have cdf given by

$$F_X(x) = \frac{1}{1+e^{-x}} \quad \text{for all } x \in \mathbb{R}.$$

Find the pdf of $X$.

## Theorem (Properties of pmfs and pdfs)

*The function* $p_x$ *is a pmf if and only if*

- $p_X(x) \geq 0$ *for all* $x \in \mathbb{R}$
- $\sum_{x \in \mathcal{X}} p_X(x) = 1.$

*The function* $f_X$ *is a pdf if and only if*

- $f_X(x) \geq 0$ *for all* $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$

**Exercise**: Consider a discrete rv $X$ with pmf given by

$$p_X(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

for some $\lambda > 0$. Show that $p_X$ is a valid pmf.

**Exercise**: Consider a continuous rv $X$ with pdf given by

$$f_X(x) = \begin{cases} \frac{1}{\lambda}e^{-x/\lambda}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

for some $\lambda > 0$. Show that $f_X$ is a valid pdf.

For a random variable $X$ with probability distribution $P_X$, we often write

- $X \sim F_X$ if $P_X$ has the cdf $F_X$
- $X \sim p_x$ if $P_X$ has the pmf $p_X$
- $X \sim f_X$ if $P_X$ has the pdf $f_X$

We can tell the support of an rv from its pdf or pmf.

The support $\mathcal{X}$ of a random variable $X$ is given by

- $\{x \in \mathbb{R} : f_X(x) > 0\}$, if $X$ is continuous with pdf $f_X$ (wherever pdf is positive)
- $\{x \in \mathbb{R} : p_X(x) > 0\}$ if $X$ is discrete with pmf $p_X$ (wherever pmf is positive).

More on finding the pdf from the cdf of a continuous rv:

- If $F_X$ has a continuous derivative $F_X'$, then $f_X = F_X'$.
- Otherwise set $f_X(x) = \frac{d}{dx} F_X(x)$ on intervals over which $F_X$ is differentiable.

**Exercise:** Let $X$ be a continuous rv with cdf given by

$$F_X(x) = \begin{cases} 0, & -\infty < x < 0 \\ 2x, & 0 \leq x < 1/3 \\ 2/3 + 1/2(x - 1/3), & 1/3 \leq x < 1 \\ 1, & 1 \leq x < \infty. \end{cases}$$

1. Draw a picture of $F_X$.
2. Find the pdf $f_X$ of the rv $X$
3. Draw a picture of $f_X$.