EXPECTED VALUE OF A RANDOM VARIABLE

Defn: The expected value $\mathbb{E}X$ of a r.v. X is

$$\mathbb{E}X = \begin{cases} \sum_{x \in \mathcal{X}} x\, p_X(x) & \text{if } X \text{ is discrete with pmf } p_X \\ \int_{-\infty}^{\infty} x\, f_X(x)\,dx & \text{if } X \text{ is continuous with density } f_X \end{cases}$$

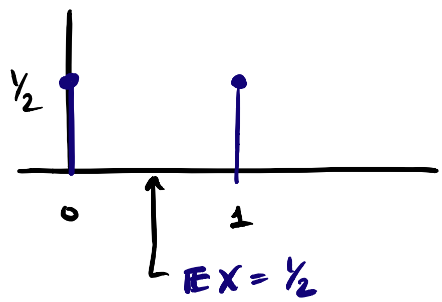
Interpretations:

- $\mathbb{E}X$ is the mean of X
- The average of many realizations of X should be close to $\mathbb{E}X$
- $\mathbb{E}X$ is the "balancing point" of the pmf/pdf

E.g. (Bernoulli r.v.) Let $X \sim p_X(x) = p^x (1-p)^{1-x}$, $x = 0, 1$.

Then $\mathbb{E}X = \sum_{x=0}^{1} x\, p^x (1-p)^{1-x} = 0 \cdot p^0 (1-p)^1 + 1 \cdot p^1 (1-p)^{1-1} = p$

For $p = 1/2$, $\mathbb{E}X = 1/2$

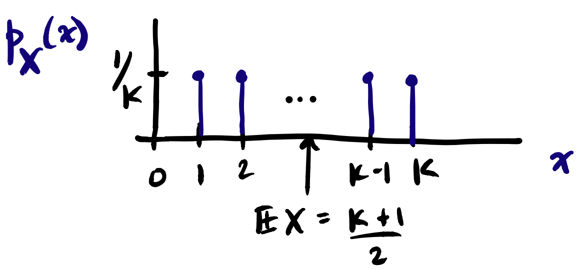


E.g. (Discrete Uniform) X = up-face of one roll of a K-sided die

Then $X \sim p_X(x) = \frac{1}{K}$, $x = 1, \ldots, K$

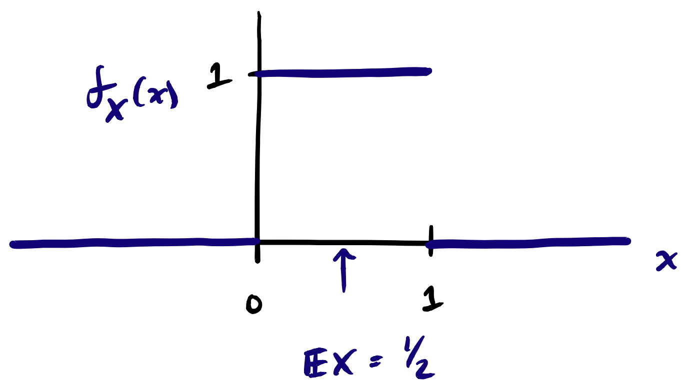
$$\mathbb{E}X = \sum_{x=1}^{K} x \cdot \frac{1}{K} = \frac{1}{K}\underbrace{(1 + \cdots + K)} = \frac{1}{K}\,\frac{K(K+1)}{2} = \frac{K+1}{2}$$

Sum of first n integers is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$



E.g. (Uniform[0,1]) X = wait time for hourly departing bus

$$f_X(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$



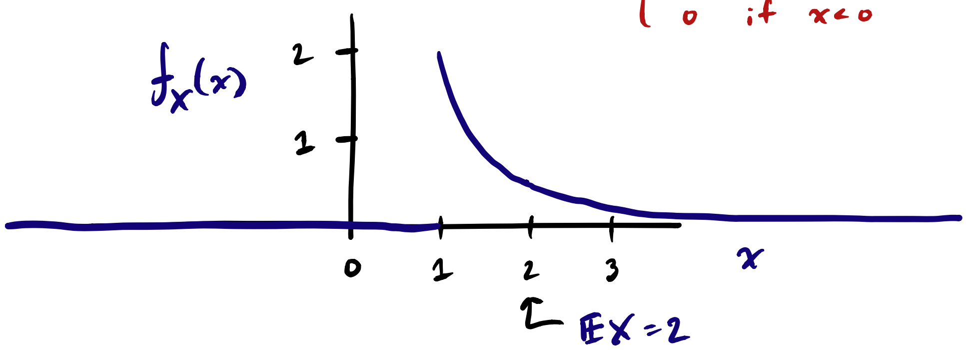
Then

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x) = \int_{-\infty}^{0} x \cdot 0 \, dx + \int_{0}^{1} x \cdot 1 \, dx + \int_{0}^{\infty} x \cdot 0 \, dx$$

$$= 0 + \frac{x^2}{2}\Big|_0^1 + 0$$

$$= 1/2$$

E.g. Let $X \sim f_X(x) = 2x^{-3} \underbrace{\mathbb{1}(x \geq 1)}_{= \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 0 \end{cases}}$



$$\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot 2x^{-3} \mathbb{1}(x \geq 1) \, dx = \int_{1}^{\infty} 2x^{-2} \, dx = \frac{2x^{-1}}{-1}\Big|_1^{\infty} = 0 - (-2) = 2$$

* We often use μ_X to denote $\mathbb{E}X$.
* We often call $\mathbb{E}X$ simply the mean of X.

Defn: The expected value of the r.v. $g(X)$, where g transforms the value of the r.v. X is

$$\mathbb{E}\, g(X) = \begin{cases} \sum_{x \in \mathcal{X}} g(x)\, p_X(x) & \text{if } X \text{ discrete with pmf } p_X \\ \int_{-\infty}^{\infty} g(x)\, f_X(x) & \text{if } X \text{ continuous with pdf } f_X \end{cases}$$

E.g. (Discrete Uniform) X = up-face of one roll of a K-sided die

Then $X \sim p_X(x) = \frac{1}{K}, \quad x = 1, \dots, K$

Let $g(x) = x^2$. Then

$$\begin{aligned} \mathbb{E}\, g(X) &= \sum_{x=1}^{K} g(x) \cdot \frac{1}{K} \\ &= \sum_{x=1}^{K} x^2 \cdot \frac{1}{K} \\ &= \frac{1}{K} \, \frac{K(K+1)(2K+1)}{6} \\ &= \frac{(K+1)(2K+1)}{6} \end{aligned}$$

Sum of first n squares

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

VARIANCE OF A RANDOM VARIABLE

Defn: The variance of a r.v. X is defined as

$$\mathrm{Var}\, X = \mathbb{E}(X - \mathbb{E}X)^2$$

$$* \quad \mathrm{Var}\, X = \mathbb{E}\, g(X) \quad \text{where} \quad g(x) = (x - \mu_X)^2$$

Interpretations:

- $\mathrm{Var}\, X$ is expected squared deviation of X from μ_X
- A measure of "spread" for the dist. of X

E.g. (Bernoulli r.v.) $X \sim p_X(x) = p^x(1-p)^{1-x}$, $x = 0, 1$.

We have $\mu_X = p$

$$\mathrm{Var}\, X = \mathbb{E}(X-p)^2 = \sum_{x=0}^{1} (x-p)^2 p^x (1-p)^{1-x}$$

$$= p^2 p^0 (1-p)^{1-0} + (1-p)^2 p^1 (1-p)^{1-1}$$

$$= p^2(1-p) + (1-p)^2 p$$

$$= (p^2 + (1-p)p)(1-p)$$

$$= p(1-p)$$

E.g. (Discrete uniform) Let $X \sim p_X(x) = \frac{1}{K}$ $x = 1, \dots, K$

We have $\mu_X = \frac{K+1}{2}$.

Left as exercise.

$$\mathrm{Var}\, X = \mathbb{E}\left(X - \frac{K+1}{2}\right)^2 = \sum_{x=1}^{K} \left(x - \frac{K+1}{2}\right)^2 \frac{1}{K} = \dots = \frac{K^2-1}{12}$$

$$\boxed{\begin{array}{l}\text{Sum of first } n \text{ squares} \\ \displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\end{array}}$$

← Must make use of this

E.g. (Uniform$[0,1]$) Let $X \sim f_X(x) = 1 \cdot \mathbb{1}(x \in [0,1])$

We have $\mu_X = 1/2$.

$$\mathrm{Var}\, X = \int_0^1 (x - 1/2)^2 \cdot 1 \, dx$$

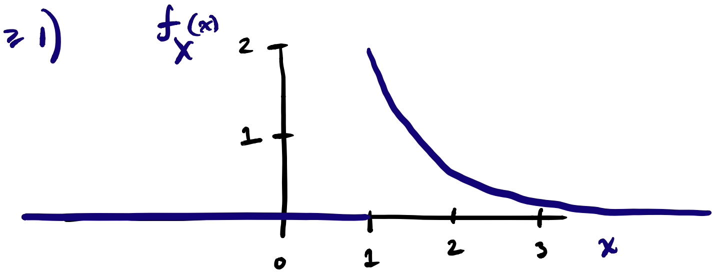
$$= \int_0^1 x^2 - x + \frac{1}{4} \, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\Big|_0^1$$

$$= \frac{1}{12}\Big[(1 - 1 + 1) - 0\Big]$$

$$= \frac{1}{12}$$

E.g. Let $X \sim f_X(x) = 2x^{-3}\,\mathbb{1}(x \geq 1)$



We have $\mu_X = 2$

$$\text{Var } X = \int_1^\infty (x-2)^2\, 2x^{-3}\, dx$$

$$= \int_1^\infty (x^2 - 4x + 4)\, 2x^{-3}\, dx$$

$$= \int_1^\infty 2x^{-1}\, dx - 4\int_1^\infty 2x^{-2}\, dx + 4\int_1^\infty 2x^{-3}\, dx$$

$$\longrightarrow 2\log(x)\Big|_1^\infty \longrightarrow \infty$$

So Var X does not exist!

SOME PROPERTIES OF THE MEAN AND VARIANCE

Theorem Let X be a r.v. with finite variance. Then for any constants a and b

(i) $\mathbb{E}(aX + b) = a\,\mathbb{E}X + b$

(ii) $\text{Var}(aX + b) = a^2\,\text{Var } X$

Proof: (i) $X \sim p_X(x)$ discrete:

$$\mathbb{E}(aX+b) = \sum_{x\in\mathcal{I}} (ax+b)\, p_X(x)$$

$$= a \sum_{x\in\mathcal{I}} x\, p_X(x) + b \sum_{x\in\mathcal{I}} p_X(x)$$

$$= a\,\mathbb{E}X + b$$

$X \sim f_X(x)$ continuous:

$$\mathbb{E}(aX+b) = \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= a\,\mathbb{E}X + b$$

(ii) $$\mathrm{Var}(aX+b) = \mathbb{E}\big(aX+b - \mathbb{E}(aX+b)\big)^2$$

$$= \mathbb{E}\big(aX+b-(a\,\mathbb{E}X+b)\big)^2$$

$$= \mathbb{E}\big(a(X-\mathbb{E}X)+b-b\big)^2$$

$$= \mathbb{E}\, a^2(X-\mathbb{E}X)^2$$

$$= a^2\, \mathbb{E}(X-\mathbb{E}X)^2$$

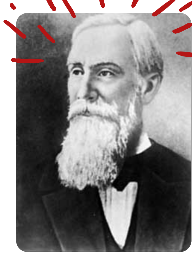
$$= a^2\, \mathrm{Var}\, X$$

Useful expression: $\mathrm{Var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2$ [Left as exercise to show this.]

Notation: We often use σ_X^2 to denote $\mathrm{Var}\, X$.

Defn: The standard deviation σ_X of a r.v. X is $\sigma_X = \sqrt{\mathrm{Var}\, X}$.

Чебышёв



Theorem (Chebychev's Inequality):

For any r.v. X with mean μ_X and variance σ_X^2 and any constant $K > 0$,

$$P_X\left(|X - \mu_X| < K\sigma_X \right) \geq 1 - \frac{1}{K^2}$$

Interp: "Any r.v. X lies within K standard deviations of its mean with probability at least $1 - 1/K^2$."

$K = 4$: Any r.v. X lies within 4 standard deviations of its mean at least 93.75 % of the time.

Proof: X discrete with pmf p_X and support $\mathcal{X}$

$$\begin{aligned}
P_X\left(|X - \mu_X| < K\sigma_X \right) &= 1 - P_X\left(|X - \mu_X| \geq K\sigma_X \right) \\
&= 1 - \sum_{\{x \in \mathcal{X} : |x - \mu_X| \geq K\sigma_X\}} p_X(x) \\
&\geq 1 - \frac{1}{K^2} \sum_{\{x \in \mathcal{X} : \left|\frac{x - \mu_X}{\sigma_X}\right| \geq K\}} \left(\frac{x - \mu_X}{\sigma_X} \right)^2 p_X(x) \\
&\geq 1 - \frac{1}{K^2} \sum_{x \in \mathcal{X}} \left(\frac{x - \mu_X}{\sigma_X} \right)^2 p_X(x) \\
&= 1 - \frac{1}{K^2}
\end{aligned}$$

X continuous with pdf f_X

$$\begin{aligned}
P_X\left(|X - \mu_X| < K\sigma_X \right) &= 1 - P_X\left(|X - \mu_X| \geq K\sigma_X \right) \\
&= 1 - \int_{\{x : |x - \mu_X| \geq K\sigma_X\}} f_X(x)\,dx
\end{aligned}$$

$$\geq 1 - \frac{1}{K^2} \int_{\left\{x : \frac{|x-\mu_X|}{\sigma_X} \geq K\right\}} \left(\frac{x-\mu_X}{\sigma_X}\right)^2 f_X(x)\,dx$$

$$\geq 1 - \frac{1}{K^2} \int_{-\infty}^{\infty} \left(\frac{x-\mu_X}{\sigma_X}\right)^2 f_X(x)\,dx$$

$$= 1 - \frac{1}{K^2} \qquad \square$$