STAT 511 fa 2019 Lec 06 slides

Expected value of a random variable

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Expected value of a random variable

The *expected value* $\mathbb{E}X$ of a random variable X is defined as

$$\mathbb{E}X = \begin{cases} \displaystyle\sum_{x \in \mathcal{X}} x \cdot p_X(x) & \text{if } X \text{ discrete with pmf } p_X \text{ and support } \mathcal{X} \\ \displaystyle\int_{-\infty}^{\infty} x \cdot f_X(x) dx & \text{if } X \text{ continuous with pdf } f_X \end{cases}$$

- The average of many realizations of X should be close to $\mathbb{E}X$.
- $\mathbb{E}X$ is the "balancing point" of the pmf/pdf.
- We often use μ_X to denote $\mathbb{E}X$.
- We often call $\mathbb{E}X$ the *mean* of X

Exercise: Let $X \sim p_X(x) = p^x(1-p)^{1-x}$ for $x = 0, 1$. Find $\mathbb{E}X$.

Exercise: Let $X =$ up-face of one roll of a K-sided die. Find $\mathbb{E}X$.

Exercise: Let $X \sim f_X(x) = \mathbf{1}(0 \leq x \leq 1)$. Find $\mathbb{E}X$.

Exercise: Let $X \sim f_X(x) = 2x^{-3}\mathbf{1}(x \geq 1)$. Find $\mathbb{E}X$.

Expected value of a function of a random variable

The expected value $\mathbb{E}g(X)$ of the rv $g(X)$, where g is any function, is

$$\mathbb{E}g(X) = \begin{cases} \displaystyle\sum_{x \in \mathcal{X}} g(x) \cdot p_X(x) & \text{if } X \text{ discrete with pmf } p_X \text{ and support } \mathcal{X} \\ \displaystyle\int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx & \text{if } X \text{ continuous with pdf } f_X \end{cases}$$

Exercise: Let X = up-face of one roll of a K-sided die. Find $\mathbb{E}X^2$.

Exercise: Let $X \sim f_X(x) = 2x^{-3}\mathbf{1}(x \geq 1)$. Find $\mathbb{E}\sqrt{X}$.

Variance of a random variable

The *variance* $\operatorname{Var} X$ of a random variable X is defined as

$$\operatorname{Var} X = \mathbb{E}(X - \mu_X)^2,$$

where $\mu_X = \mathbb{E}X$.

- $\operatorname{Var} X$ is the expected squared deviation of X from μ_X.
- Measure of "spread" for the distribution of X.
- Often use σ_X^2 to denote $\operatorname{Var} X$.
- Use σ_X to denote $\sqrt{\operatorname{Var} X}$, which is called the *standard deviation* of X.

Useful expression: $\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

Exercise: Let $X \sim p_X(x) = p^x(1-p)^{1-x}$ for $x = 0, 1$. Find $\operatorname{Var} X$.

Exercise: Let $X \sim f_X(x) = \mathbf{1}(0 \leq x \leq 1)$. Find $\operatorname{Var} X$.

Theorem (Mean and variance of shifted and scaled random variables)

Let X be an rv with finite mean and variance. Then for any constants a and b

- $\mathbb{E}(aX + b) = a\mathbb{E}X + b$
- $\operatorname{Var}(aX + b) = a^2 \operatorname{Var} X$

Exercise: Let $X \sim f_X(x) = \mathbf{1}(0 \leq x \leq 1)$ and suppose $Y = 4X - 2$.

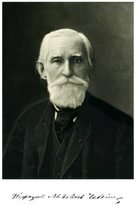
1. $\mathbb{E}Y$.
2. $\operatorname{Var} Y$.

Theorem (Чебышёв's inequality)

For any rv X with mean μ_X and var. $\sigma_X^2 < \infty$ and any constant $K > 0$, we have

$$P_X(|X - \mu_X| < K\sigma_X) \geq 1 - \frac{1}{K^2}.$$

- Any rv X lies within K st'd dev's of its mean with prob. at least $1 - 1/K^2$.
- E.g. any rv X lies within 4 st'd dev's of its mean at least 93.75% of the time (since $1 - 1/4^2 = 0.9375$).



Prove the result.