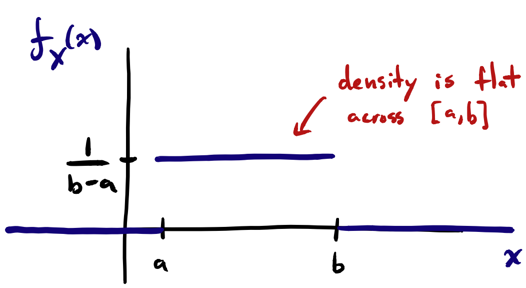
SUITE OF OUGHT-TO-KNOW PROBABILITY DISTRIBUTIONS FOR CONTINUOUS RANDOM VARIABLES

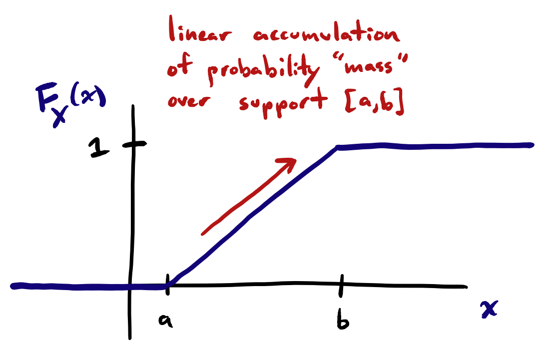
* optional

(i) Uniform$[a,b]$: When X takes any values in $[a,b]$ with equal probability.

$$f_X(x;a,b) = \frac{1}{b-a}\mathbb{1}(a\le x\le b)$$

$$F_X(x;a,b) = \begin{cases} 1 & b\le x<\infty \\ \frac{x-a}{b-a} & a\le x<b \\ 0 & -\infty<x<a \end{cases}$$





$\mathbb{E}X = ?$

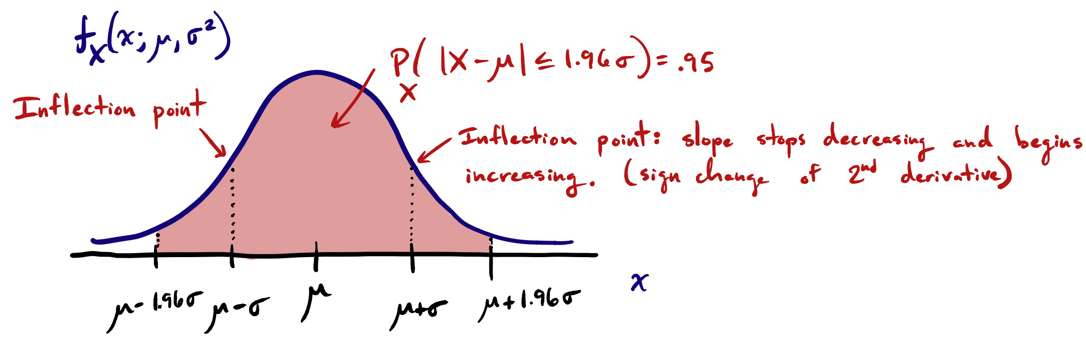
$\text{Var }X = ?$

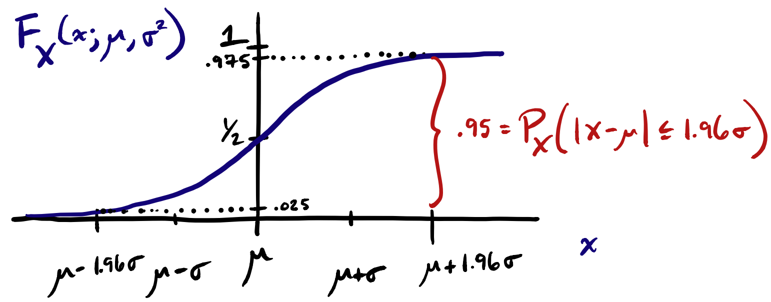
Left as exercise

(ii) **Normal(μ, σ^2)** That ubiquitous bell-shaped curve. Very important...

$$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$F_X(x;\mu,\sigma^2) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right]dt$$ (No nice simplification available)





$\mathbb{E}X = \mu$

$\text{Var}\, X = \sigma^2$

* **Special case:** Normal$(\underset{\mu=0}{0}, \underset{\sigma^2=1}{1})$ called Standard Normal distribution

Normal$(0,1)$ pdf and cdf get special notation:

pdf: $\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]$

Greek letter "phi"

cdf: $\Phi(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{t^2}{2}\right]dt$

Capital phi

Result: If $X \sim \text{Normal}(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, then $Z \sim \text{Normal}(0,1)$.

Proof: Idea of proof: Let $X \sim \text{Normal}(\mu,\sigma^2)$ and show that the cdf of $Z = (X-\mu)/\sigma$ is Φ.

$$P_Z(Z \le z) = P_X\left(\frac{X-\mu}{\sigma} \le z\right)$$

big Z is the random variable Z

little z, just some value

$$= P_X(X \le \mu + \sigma z)$$

$$= F_X(\mu + \sigma z; \mu, \sigma)$$

$$= \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] dt$$

set $u = \frac{t-\mu}{\sigma}$, $t = \mu + \sigma u$

so $\frac{du}{dt} = \frac{1}{\sigma}$, $dt = \sigma\, du$

and $-\infty < t < \mu + \sigma z$, $-\infty < u < z$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du$$

$$= \Phi(z) \qquad \square$$

Mean and variance of Normal(0,1): $\mathbb{E}Z = 0$ and $\text{Var}\, Z = 1$.

Show $\mathbb{E}Z = 0$:

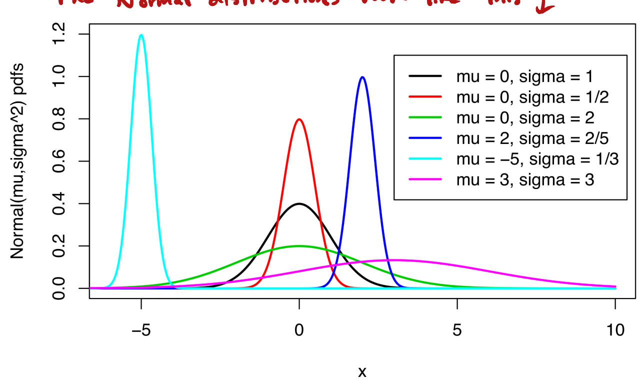
$$\mathbb{E}Z = \int_{-\infty}^{\infty} z\,\phi(z)\,dz = \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz = -\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right]\Bigg|_{-\infty}^{\infty} = 0$$

Show $\text{Var}\, Z = 1$:

$$\text{Var}\, Z = \mathbb{E}Z^2 - (\underbrace{\mathbb{E}Z}_{0})^2 = \mathbb{E}Z^2,$$

$$\mathbb{E}Z^2 = \int_{-\infty}^{\infty} z^2 \phi(z)\, dz$$

The Normal distributions look like this:



$$= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz$$

$$= \int_{-\infty}^{\infty} \underbrace{z}_{v} * \underbrace{z \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz}_{du} \qquad u = -\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right]$$

$$dv = dz$$

Integration by parts

$$= -z \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right]\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz$$

Integration over Normal(0,1) pdf

$$= 1$$

So $\operatorname{Var} Z = 1$.

Integration by parts:

$$\frac{d}{dx} u(x)v(x) = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

$$\Leftrightarrow u(x)v(x) = \int u(x)\frac{d}{dx}v(x)\,dx + \int v(x)\frac{d}{dx}u(x)\,dx$$

$$\Leftrightarrow \int v(x)\frac{d}{dx}u(x)\,dx = u(x)v(x) - \int u(x)\frac{d}{dx}v(x)\,dx$$

Establishing $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$:

Is $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right]$ a legitimate pdf? (does it integrate to 1?)

We have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \underbrace{\exp\left[-\frac{z^2}{2}\right]}_{\text{No nice antiderivative for } \exp[-z^2/2]} dz = \underbrace{2}_{\text{By symmetry around } z=0} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz = \sqrt{\frac{2}{\pi}} \underbrace{\int_0^{\infty} \exp\left[-\frac{z^2}{2}\right] dz}_{\text{We need to show that this is equal to } \sqrt{\frac{\pi}{2}}.}$$

It is sufficient to show $\left(\int_0^{\infty} \exp\left[-\frac{z^2}{2}\right] dz\right)^2 = \frac{\pi}{2}$. We have

$$\left(\int_0^{\infty} \exp\left[-\frac{z^2}{2}\right] dz\right)^2 = \left(\int_0^{\infty} \exp\left[-\frac{t^2}{2}\right] dt\right)\left(\int_0^{\infty} \exp\left[-\frac{u^2}{2}\right] du\right)$$

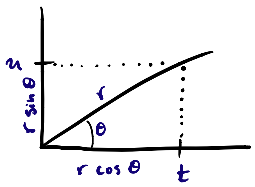
$$= \int_0^\infty \int_0^\infty \exp\left[-\frac{t^2+u^2}{2}\right] dt\,du$$

Now it gets tricky: switch to polar coordinates by

[See Early Transcendentals Calculus, James Stewart 5th ed, pg. 1005]

$$t = r\cos\theta$$

$$u = r\sin\theta$$



$0 \le \theta \le \pi/2$, since $t \ge 0$, $u \ge 0$.

Then $t^2 + u^2 = r^2$ and $dt\,du = r\,dr\,d\theta$, so

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{t^2+u^2}{2}\right] dt\,du = \int_0^{\pi/2} \int_0^\infty r \exp\left[-\frac{r^2}{2}\right] dr\,d\theta$$

$$= \int_0^{\pi/2} \left(-\exp\left[-\frac{r^2}{2}\right]\right)\Bigg|_0^\infty d\theta$$

$$= \int_0^{\pi/2} 1\,d\theta$$

$$= \frac{\pi}{2}.$$

So we have shown that $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$.

Showing $\mathbb{E}X = \mu$ and $\operatorname{Var} X = \sigma^2$:

Let $X \sim \text{Normal}(\mu, \sigma^2)$ and $Z \sim \text{Normal}(0,1)$.
Then X and $\mu + \sigma Z$ have the same probability distribution, so

$$\mathbb{E}X = \mathbb{E}[\mu + \sigma Z] = \mu + \sigma\,\mathbb{E}Z = \mu$$

$$\operatorname{Var} X = \operatorname{Var}[\mu + \sigma Z] = \sigma^2 \operatorname{Var} Z = \sigma^2$$

(iii) **Gamma(α, β)**: Family of right-skewed distributions with support on $(0,\infty)$.

Based on the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp[-x]\, dx \qquad \text{for} \quad \alpha > 0.$$

Special properties of gamma function:

(a) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for all $\alpha > 0$

(b) For any integer $n > 0$, $\Gamma(n) = (n-1)!$

(c) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Observe: $\frac{1}{\Gamma(\alpha)} x^{\alpha-1} \exp[-x]\, \mathbb{1}(x>0)$ is a pdf, since

$$\int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \exp[-x]\, dx = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) = 1.$$

The full gamma family has an additional parameter β built in:

α called the "shape" parameter

$$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left[-\frac{x}{\beta}\right] \mathbb{1}(x>0)$$

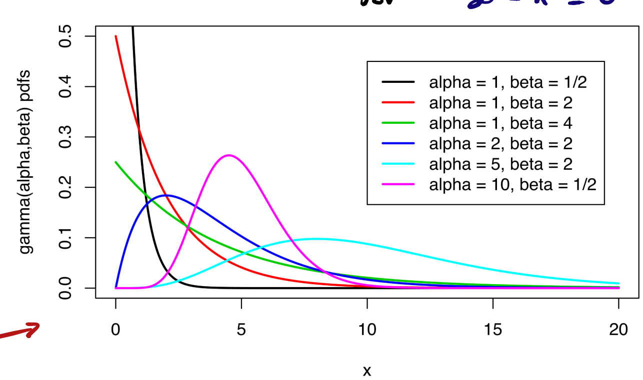
β called the "scale" parameter

$$F_X(x;\alpha,\beta) = \begin{cases} \int_0^x \frac{1}{\Gamma(\alpha)\beta^\alpha} t^{\alpha-1} \exp\left[-\frac{t}{\beta}\right] dt & \text{for} \quad 0 < x < \infty \\ 0 & \text{for} \quad -\infty < x \le 0 \end{cases}$$

$\mathbb{E}X = \alpha\beta$

$\mathrm{Var}\, X = \alpha\beta^2$

Gamma distributions look like this



Derivations

(a) Show $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$ for all $\alpha > 0$.

$$\Gamma(\alpha) = \int_0^\infty \underbrace{x^{\alpha-1} \exp[-x]}_{v}\, \underbrace{dx}_{du}$$

$u = x$

$dv = \Big((\alpha-1)x^{\alpha-2}\exp[-x] - x^{\alpha-1}\exp[-x]\Big)dx$

Integration by parts

$$= \underbrace{x \cdot x^{\alpha-1}\exp[-x]\Big|_0^\infty}_{=0} - \int_0^\infty x \cdot \Big((\alpha-1)x^{\alpha-2}\exp[-x] - x^{\alpha-1}\exp[-x]\Big)\,dx$$

$$= -\int_0^\infty (\alpha-1)\,x^{\alpha-1}\exp[-x] + \int x^{(\alpha+1)-1}\exp[-x]\,dx$$

$$= (1-\alpha)\,\Gamma(\alpha) + \Gamma(\alpha+1)$$

and

$$\Gamma(\alpha) = (1-\alpha)\,\Gamma(\alpha) + \Gamma(\alpha+1)$$

$$\Longleftrightarrow$$

$$\alpha\,\Gamma(\alpha) = \Gamma(\alpha+1).$$

(b) Show that for any integer $n > 0$, $\Gamma(n) = (n-1)!$

First: $\Gamma(1) = \int_0^\infty x^{1-1}\exp[-x] = -\exp[-x]\Big|_0^\infty = 1$

So

$$\left.\begin{aligned}\Gamma(n) &= (n-1)\,\Gamma(n-1)\\ &= (n-1)(n-2)\,\Gamma(n-2)\\ &\;\;\vdots\\ &= (n-1)(n-2)\cdot\ldots\cdot\Gamma(1)\end{aligned}\right\}\ \text{because } \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \text{ for all } \alpha > 0$$

$$= (n-1)(n-2)\cdot\ldots\cdot 1$$

$$= (n-1)!$$

(c) Show $\Gamma(1/2) = \sqrt{\pi}$:

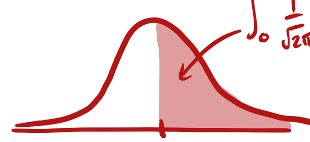
$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{1/2 - 1} \exp[-x]\, dx$$

set $y = \sqrt{2x} \Rightarrow x = \frac{y^2}{2}$, $dx = y$

$$= \int_0^\infty \left[\frac{y^2}{2}\right]^{1/2 - 1} \exp\left[-\frac{y^2}{2}\right] y\, dy$$

$$= \sqrt{2} \int_0^\infty \exp\left[-\frac{y^2}{2}\right] dy$$

$$= \sqrt{2}\sqrt{2\pi} \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-y^2\right] dy}_{\text{integral over one half of Normal(0,1) pdf}}$$



Divide by 2 and integrete from $-\infty$ to ∞

$$= \sqrt{\pi} \underbrace{\int_{-\infty}^\infty \phi(y)\, dy}_{=1}$$

$$= \sqrt{\pi}$$

Show $\mathbb{E}X = \alpha\beta$

$$\mathbb{E}X = \int_0^\infty x \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha - 1} \exp\left[-\frac{x}{\beta}\right] dx$$

$$= \alpha\beta \int_0^\infty \frac{1}{\alpha\Gamma(\alpha)\, \beta^{\alpha+1}} x^\alpha \exp\left[-\frac{x}{\beta}\right] dx$$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$$= \alpha\beta \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha+1)\, \beta^{\alpha+1}} x^{(\alpha+1)-1} \exp\left[-\frac{x}{\beta}\right] dx}_{=1\text{, integral over gamma}(\alpha+1,\beta)\text{ pdf}}$$

$$= \alpha\beta$$

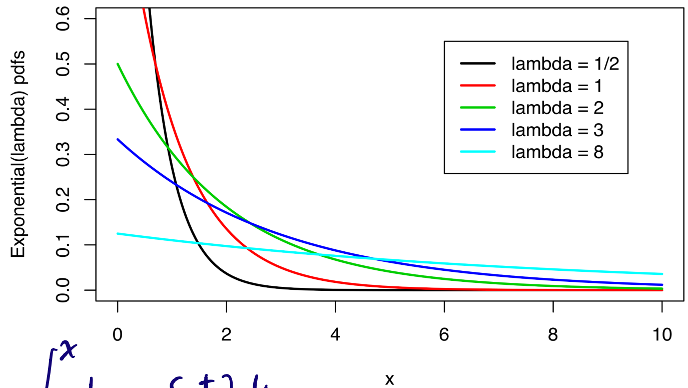
Show $\mathrm{Var}\, X = \alpha\beta^2$ (Left as exercise: find $\mathbb{E}X^2$ similarly and use $\mathrm{Var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2$)

(iv) **Exponential(λ):** Highly right-skewed with support on $(0, \infty)$.

$$f_X(x;\lambda) = \frac{1}{\lambda}\exp\left[-\frac{x}{\lambda}\right]\mathbb{1}(x>0)$$

Exponential distributions look like this

$$F_X(x;\lambda) = \begin{cases} 1-\exp\left[-\frac{x}{\lambda}\right] & 0<x<\infty \\ 0 & -\infty < x \leq 0 \end{cases}$$



$\mathbb{E}X = \lambda$

$\operatorname{Var} X = \lambda^2$

Derivations:

Get the cdf $F_X(x;\lambda)$:

For $x \geq 0$,
$$F_X(x;\lambda) = \int_0^x \frac{1}{\lambda}\exp\left[-\frac{t}{\lambda}\right]dt$$
$$= -\exp\left[-\frac{t}{\lambda}\right]\Big|_0^x$$
$$= 1-\exp\left[-\frac{x}{\lambda}\right].$$

Show $\mathbb{E}X = \lambda$ and $\operatorname{Var} X = \lambda^2$:

Exponential(λ) is same as Gamma(α, β) with $\alpha = 1$, $\beta = \lambda$.

$\mathbb{E}X = \alpha\beta$, $\operatorname{Var} X = \alpha\beta^2$

We will later call this parameter the "degrees of freedom."

(v) **Chi-squared(ν):** Right skewed with support on $(0,\infty)$.

$$f_X(x;\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}}x^{\nu/2-1}\exp\left[-\frac{x}{2}\right]\mathbb{1}(x>0)$$

Greek letter "nu"

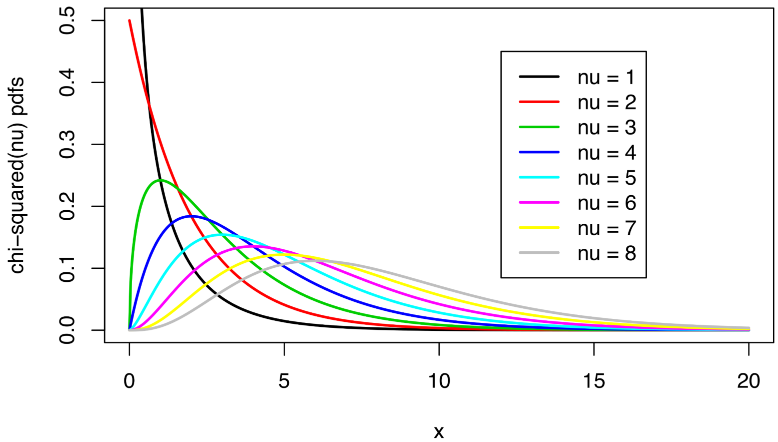
$$F_X(x;\nu) = \begin{cases} \int_0^x \frac{1}{\Gamma(\nu/2)2^{\nu/2}}t^{\nu/2-1}\exp\left[-\frac{t}{2}\right]dt & \text{for } 0<x<\infty \\ 0 & \text{for } -\infty<x\leq 0 \end{cases}$$

$\mathbb{E}X = \nu$

$\mathrm{Var}\,X = 2\nu$

Chi-squared(ν) is same as Gamma(α,β) with $\alpha = \nu/2$, $\beta = 2$

Chi-squared distributions look like this →



(vi) **Beta(α,β)**: Family of distributions with support on $(0,1)$.

Based on the beta function:

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx, \quad \text{for } \alpha>0,\ \beta>0.$$

It turns out $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Thus $\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}(0<x<1)$ integrates to 1, i.e., is a pdf.

So:

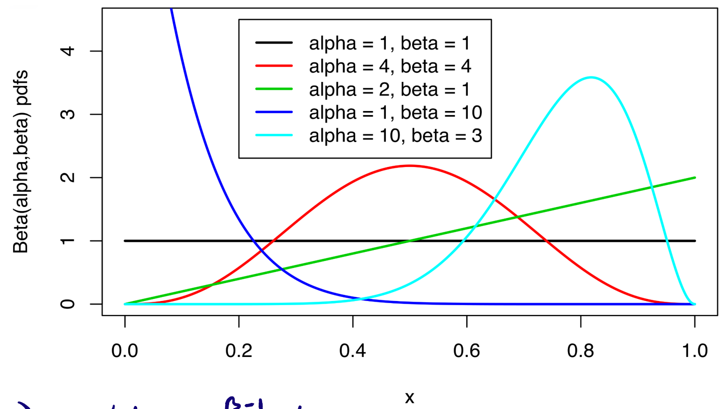
$$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}(0<x<1)$$

$$F_X(x;\alpha,\beta) = \begin{cases} 1 & \text{for } 1\le x<\infty \\ \displaystyle\int_0^x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} & \text{for } 0<x<1 \\ 0 & \text{for } -\infty<x\le 0 \end{cases}$$

$$\mathbb{E}X = \frac{\alpha}{\alpha+\beta}$$

$$\mathrm{Var}\,X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Beta distributions look like this ↓



Derivations

Show $\mathbb{E}X = \frac{\alpha}{\alpha+\beta}$

$$\mathbb{E}X = \int_0^1 x \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\alpha}{\alpha+\beta}\int_0^1 \frac{(\alpha+\beta)}{\alpha}\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha)} x^{(\alpha+1)-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\alpha}{\alpha+\beta}\underbrace{\int_0^1 \frac{\Gamma((\alpha+1)+\beta)}{\Gamma(\beta)\,\Gamma(\alpha+1)} x^{(\alpha+1)-1}(1-x)^{\beta-1}\,dx}_{=1 \text{ Integral over Beta}(\alpha+1,\beta)\text{ pdf}}$$

$$= \frac{\alpha}{\alpha+\beta}$$

Show $\mathrm{Var}\,X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ ← Left as exercise. Find $\mathbb{E}X^2$ similarly and use $\mathrm{Var}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2$.

* When $\alpha=1, \beta=1$, we get the Uniform$(0,1)$ distribution.

(vii)* Normal Mixture$\left((\mu_k, \sigma_k^2, \alpha_k)_{k=1}^m\right)$

(viii)* KDE$(x_1,\dots x_n, K, h)$

Potentially to be added to these notes...