STAT 511 su 2020 Lec 08 slides

Suite of ought-to-know probability distributions of continuous random variables

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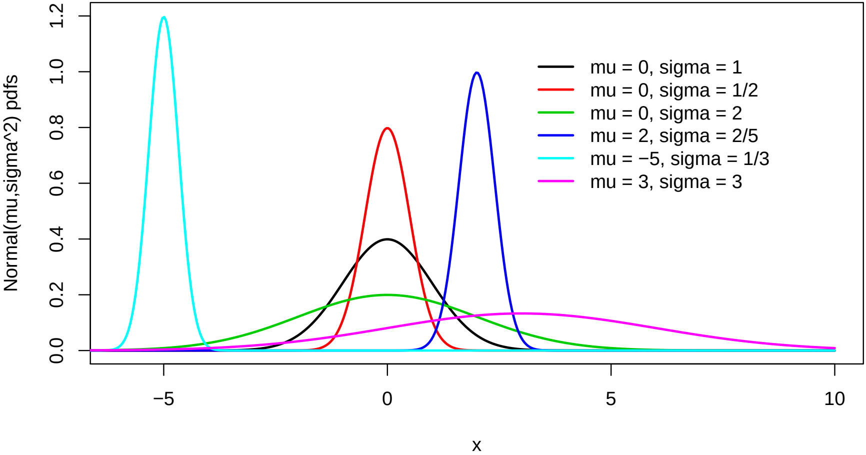
These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

- The pdf of the Uniform(a, b) distribution is given by

$$f_X(x; a, b) = \frac{1}{b-a}\mathbf{1}(a < x < b) \quad \text{for } x \in \mathbb{R},$$

  for $a < b$.
- Parameters:
  - a is the lower bound of the support
  - b is the upper bound of the support
- The Uniform$(0, 1)$ pdf is $f_X(x) = \mathbf{1}(0 < x < 1)$.

pdfs of several Normal distributions



- The pdf of the Normal(μ, σ^2) distribution is given by

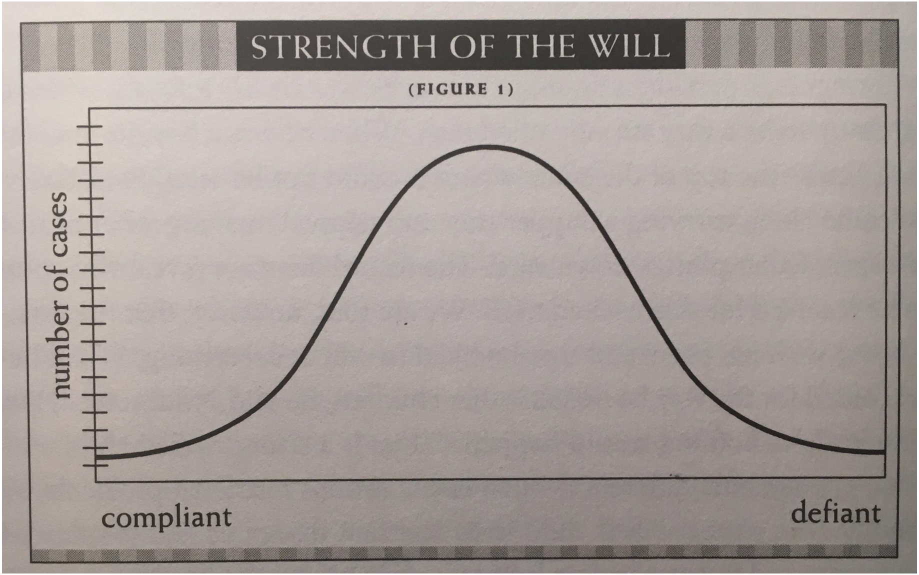
$$f_X(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad \text{for } x \in \mathbb{R}.$$

- Parameters:
  - μ is a *location parameter*
  - σ is a *scale parameter*
- If $X \sim \text{Normal}(\mu, \sigma^2)$, then
  - $\mathbb{E}X = \mu$
  - $\text{Var}\, X = \sigma^2$
- The pdf and cdf of the Normal$(0, 1)$ distribution get special notation:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \quad \text{for } z \in \mathbb{R}.$$

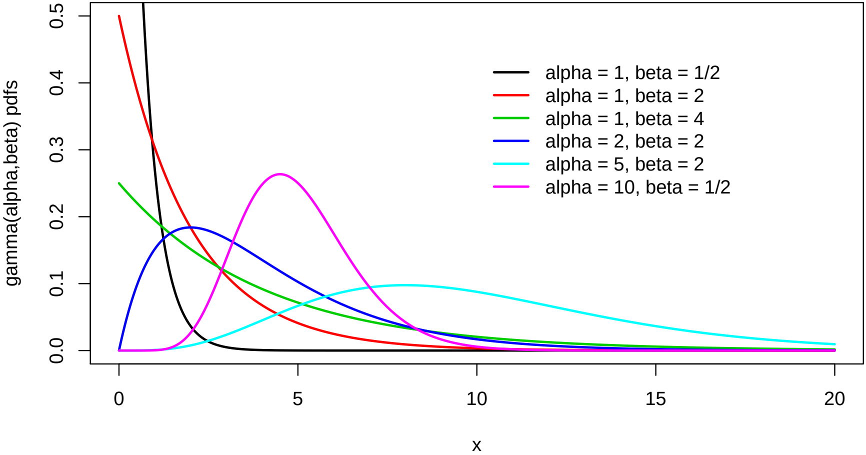
Example of popular usage of the Normal distribution; see [1].



Exercises: Establish the following:

1. If $X \sim \text{Normal}(\mu, \sigma^2)$ then $Z = (X - \mu)/\sigma \sim \text{Normal}(0, 1)$.
2. $\mathbb{E}Z = 1$.
3. $\text{Var}\, Z = 1$.
4. $\int_{-\infty}^{\infty} \phi(z)dz = 1$.
5. $\mathbb{E}X = \mu$.
6. $\text{Var}\, X = \sigma^2$.

pdfs of several Gamma distributions



- The pdf of the Gamma(α, β) distribution is given by

$$f_X(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left[-\frac{x}{\beta}\right] \quad \text{for } x > 0.$$

- Parameters:
  - α is a *shape parameter*
  - β is a *scale parameter*
- If $X \sim \text{Gamma}(\alpha, \beta)$, then
  - $\mathbb{E}X = \alpha\beta$
  - $\text{Var}\, X = \alpha\beta^2$

The gamma distributions are brought to you by the gamma function.

Gamma function

For any $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, the *gamma function* is given by

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du.$$

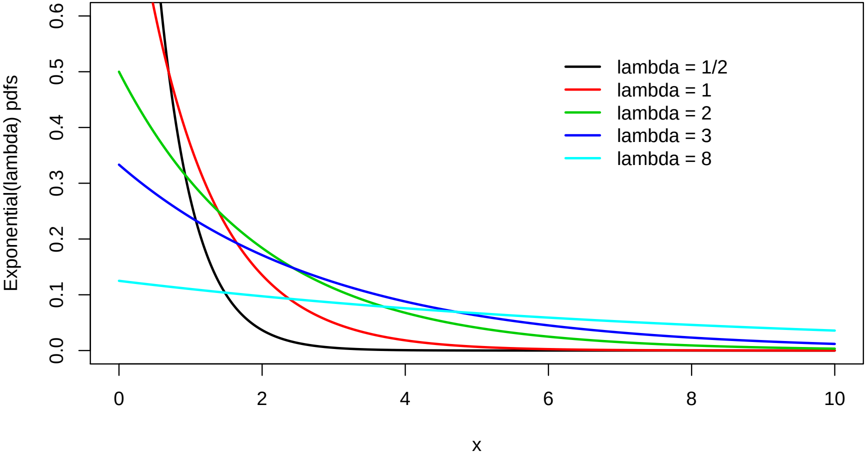
These are some of its properties:

1. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for all $\text{Re}(\alpha) > 0$.
2. $\Gamma(n) = (n-1)!$ for any integer $n > 0$.
3. $\Gamma(1/2) = \sqrt{\pi}$.

Exercise:

1. Prove the above properties.
2. Show $\mathbb{E}X = \alpha\beta$ if $X \sim \text{Gamma}(\alpha, \beta)$.

pdfs of several Exponential distributions



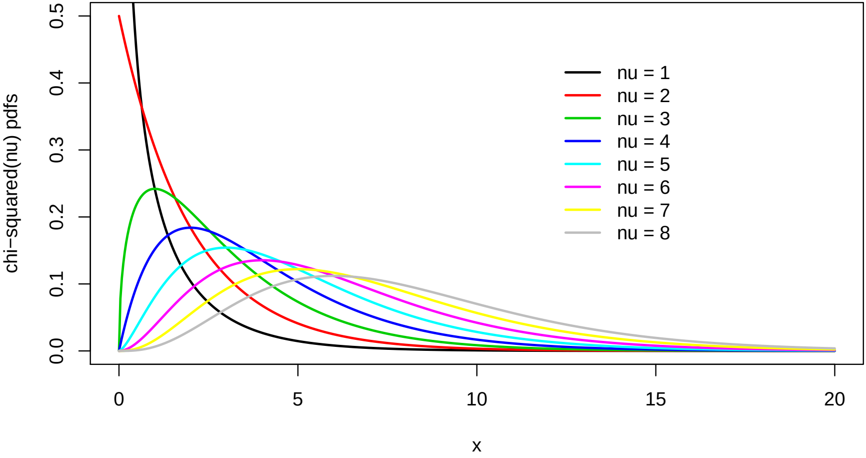
- The pdf of the Exponential(λ) distribution is given by

$$f_X(x;\lambda) = \frac{1}{\lambda}\exp\left[-\frac{x}{\lambda}\right] \qquad \text{for } x > 0.$$

- Parameter:
  - λ is a *scale parameter*
- If $X \sim \text{Exponential}(\lambda)$, then
  - $\mathbb{E}X = \lambda$
  - $\text{Var}\, X = \lambda^2$

Exercise: Find the cdf of the Exponential(λ) distribution.

pdfs of several Chi-squared distributions

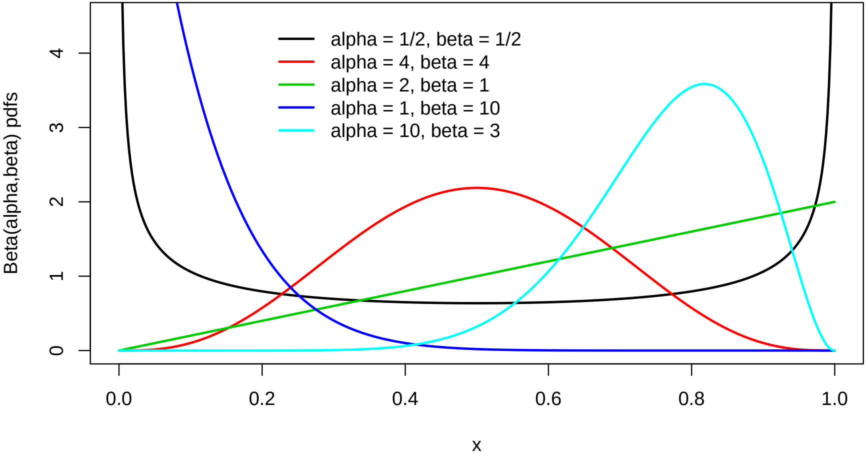


- The pdf of the Chi-squared(ν) distribution is given by

$$f_X(x;\nu) = \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2-1} \exp\left[-\frac{x}{2}\right] \quad \text{for } x > 0.$$

- Parameter:
  - ν is called the *degrees of freedom*
- If $X \sim$ Chi-squared(ν), then
  - $\mathbb{E}X = \nu$
  - $\text{Var}\, X = 2\nu$

pdfs of several Beta distributions



- The pdf of the Beta(α, β) distribution is given by

$$f_X(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } x \in (0,1).$$

- Parameter:
  - α is a *shape parameter*
  - β is a *shape parameter*
- If $X \sim \text{Beta}(\alpha, \beta)$, then
  - $\mathbb{E}X = \dfrac{\alpha}{\alpha+\beta}$
  - $\text{Var}\, X = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- The Beta$(1,1)$ distribution is the Uniform$(0,1)$ distribution.

The beta distributions are brought to you by the beta function.

Beta function

For any $\alpha, \beta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$, the *beta function* is given by

$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Exercise: Show $\mathbb{E}X = \alpha/(\alpha+\beta)$ if $X \sim \text{Beta}(\alpha, \beta)$.

James C Dobson.
The new strong-willed child.
Tyndale House Publishers, Inc., 2017.