STAT 511 su 2020 Lec 10 slides

Moments and moment-generating functions

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Moments about the origin and about the mean

Let X be a random variable. For each integer k, the

- *kth moment about the origin* of X is

$$\mu_k' = \mathbb{E}X^k$$

- *kth moment about the mean* of X is

$$\mu_k = \mathbb{E}(X - \mu)^k,$$

where $\mu := \mu_1'$.

Also refer to moments about the mean as *central moments*.

Usually refer to moments about the origin simply as *moments*.

$\text{Var}\, X = \mu_2 = \mu_2' - (\mu_1')^2 =$ the 2nd moment minus the square of the 1st moment

Moment-generating function

The *moment-generating function (mgf)* M_X of a rv X is the function given by

$$M_X(t) = \mathbb{E}\exp(tX),$$

provided the expectation is finite for t in a neighborhood of 0.

"A moment-generating function is a function that generates moments."

–Dr. Josh Tebbs

Theorem (generating moments with the moment-generating function)

If X is a rv with mgf M_X, then

$$\mathbb{E}X^k = M_X^{(k)}(0),$$

where

$$M_X^{(k)}(0) = \left(\frac{d}{dt}\right)^k M_X(t)\Big|_{t=0}.$$

Recipe: To get the kth moment we

1. differentiate the mgf k times with respect to t
2. evaluate the result at $t = 0$.

Prove the result.

Exercise: Let $X \sim \text{Exponential}(\lambda)$.

1. Find the mgf of X.
2. Use the mgf of X to find $\mathbb{E}X$ and $\text{Var}\,X$.

Exercise: Let $X \sim \text{Binomial}(n, p)$.

1. Find the mgf of X.
2. Use the mgf of X to find $\mathbb{E}X$ and $\text{Var}\, X$.

Mgfs really more useful for characterizing distributions than for getting moments.

Theorem (identification of distribution by mgf)

Let X and Y be rvs such that $\mathbb{E}X^k < \infty$ and $\mathbb{E}Y^k < \infty$ for all $k = 1, 2, \dots,$ and

- *M_X and M_Y exist*
- *$M_X(t) = M_Y(t)$ for all t in a neighborhood of 0*

Then X and Y are identically distributed.

So rvs with the same mgf have the same distribution; the mgf identifies the dist.

We will later use mgfs to prove a version of the Central Limit Theorem!

Exercise: Let $Z \sim \text{Normal}(0, 1)$. Find the mgf of Z.

Mgfs also help us find the distributions of linearly transformed rvs.

Btw a *linear transformation* is any shift-and-scale transformation.

Theorem (mgf of shifted and scaled random variable)

Let X be a rv with mgf M_X. For any constants $a, b \in \mathbb{R}$, the mgf of $aX + b$ is

$$M_{aX+b}(t) = e^{tb} M_X(at).$$

Exercise: Let $Z \sim \text{Normal}(0, 1)$ and let $X = \sigma Z + \mu$. Find the mgf of X.

Exercise: Let $X \sim \text{Exponential}(10)$. Find the distribution of $Y = 2X$.