CONDITIONAL DISTRUBUTIONS

Let (X,Y) be a pair of discrete r.v.s with support $\mathcal{X}$ and $\mathcal{Y}$, respectively, joint pmf p and marginal pmfs p_X and p_Y.

Consider the events $\{X \in A\}$ and $\{Y \in B\}$ for some $A \subset \mathcal{X}$ and $B \subset \mathcal{Y}$.

The conditional probability of $\{X \in A\}$ given $\{Y \in B\}$ is given by

$$P(X \in A \mid Y \in B) = \frac{P(\{X \in A\} \cap \{Y \in B\})}{P_Y(Y \in B)}$$

$$= \frac{P(\{s \in S : (X(s), Y(s)) \in A \times B\})}{P(\{s \in S : Y(s) \in B\})}.$$

We may now define the conditional pmf of X given Y:

Defn: For any value y for which $p_Y(y) > 0$, the conditional pmf of X given that $Y = y$ is the function given by

$$p(x|y) = P(X = x \mid Y = y) = \frac{\overbrace{p(x,y)}^{\text{joint mf of } (X,Y)}}{\underbrace{p_Y(y)}_{\text{marginal pmf of } Y}}$$

for all $x \in \mathbb{R}$.

Likewise, for any value x for which $p_X(x) > 0$, the conditional pmf of Y given that $X = x$ is the function given by

$$p(y|x) = P(Y = y \mid X = x) = \frac{p(x,y)}{\underbrace{p_X(x)}_{\text{marginal pmf of } X}}$$

for all $y \in \mathbb{R}$.

Remark: The conditional pmfs are legitimate pmfs:

- $$\sum_{x \in \mathcal{X}} p(x|y) = \sum_{x \in \mathcal{X}} \frac{p(x,y)}{p_Y(y)} = \frac{p_Y(y)}{p_Y(y)} = 1$$

- $p(x|y) \geq 0$ for all $x \in \mathbb{R}$ since $p(x,y) \geq 0$ and $p_Y(y) > 0$.

  (likewise for $p(y|x)$)

E.g. Roll two dice. Let X = sum of rolls, Y = maximum of rolls.

Get conditional pmfs of X given $Y=4$ and Y given $X=7$.

Consider single column or single row of joint pmf table:

$x \backslash y$	1	2	3	4	5	6	$p_X(x)$
2	1/36						1/36
3		2/36					2/36
4		1/36	2/36				3/36
5			2/36	2/36			4/36
6			1/36	2/36	2/36		5/36
7				2/36	2/36	2/36	6/36
8				1/36	2/36	2/36	5/36
9					2/36	2/36	4/36
10					1/36	2/36	3/36
11						2/36	2/36
12						1/36	1/36
$p_Y(y)$	1/36	3/36	5/36	7/36	9/36	11/36	1

← marginal pmf of X

Joint probabilities $p(x,y)$

marginal pmf of Y

Conditional pmf of X given that $Y=4$

x	$p(x\|4)$
5	2/7
6	2/7
7	2/7
8	1/7

$$2/7 = \frac{p(5,4)}{p_Y(4)} = \frac{2/36}{7/36}$$

y	4	5	6
$p(y\|7)$	2/6	2/6	2/6

Conditional pmf of Y given that $X=7$

Let (X,Y) be a pair of continuous r.v.s with joint pdf f and marginal pdfs f_X and f_Y.

We define conditional pdfs much like we have defined conditional pmfs.

Defn: For any $y \in \mathbb{R}$ such that $f_Y(y) > 0$, the conditional pdf of X given that $Y=y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{for all } x \in \mathbb{R}.$$

Likewise, for any $x \in \mathbb{R}$ such that $f_X(x) > 0$, the conditional pdf of Y given that $X=x$ is

$$f(y|x) = \frac{f(x,y)}{f_X(x)} \quad \text{for all } y \in \mathbb{R}.$$

Remark: The conditional pdfs are legitimate pdfs:

- $$\int_{\mathbb{R}} f(x|y)\,dx = \int_{\mathbb{R}} \frac{f(x,y)}{f_Y(y)}\,dx = \frac{f_Y(y)}{f_Y(y)} = 1$$
- $f(x|y) \geq 0$ for all $x \in \mathbb{R}$ since $f(x,y) \geq 0$ and $f_Y(y) > 0$.

(Likewise for $f(y|x)$)

Eg. Let (X,Y) have joint pdf given by

$$f(x,y) = y(1-x)^{y-1} e^{-y} \mathbb{1}(0 < x < 1, 0 < y < \infty).$$

(a) Find the conditional pdf $f(x|y)$ of $X|Y=y$.

We have

$$f(x|y) = \frac{f(x,y)}{f_Y(y)},$$

So we first need to find the marginal pdf of Y:

$$f_Y(y) = \int_0^1 y(1-x)^{y-1} e^{-y}\, dx \cdot \mathbb{1}(0<y<\infty)$$

$$= y e^{-y} \cdot \mathbb{1}(0<y<\infty) \cdot \underbrace{\int_0^1 (1-x)^{y-1}\, dx}_{=\frac{1}{y}}$$

$$= e^{-y} \cdot \mathbb{1}(0<y<\infty).$$

So, for $0<x<1$ and $y>0$, we have

$$f(x|y) = \frac{y(1-x)^{y-1} e^{-y}}{e^{-y}}$$

$$= y(1-x)^{y-1}.$$

If we study this pdf carefully, we can see that

$$X|Y=y \sim \text{Beta}(1, y).$$

(b) Find $P(X<\frac{1}{2} \mid Y=2)$.

We have

$$P\left(X<\tfrac{1}{2} \mid Y=2\right) = \int_0^{1/2} 2(1-x)^{2-1}\, dx$$

$$= 2\left(x - \frac{x^2}{2}\right)\Bigg|_0^{1/2}$$

$$= 2\left(\frac{1}{2} - \frac{1}{4}\cdot\frac{1}{2}\right)$$

$$= \frac{3}{4}.$$

Eg: Let (X, Y) have joint pdf given by

$$f(x,y) = \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2+1)\right] \mathbb{1}(0 < x < \infty, -\infty < y < \infty).$$

(a) Find the conditional pdf $f(y|x)$ of $Y | X = x$.

Since

$$f(y|x) = \frac{f(x,y)}{f_X(x)},$$

we must first find the marginal pdf f_X of X.

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2+1)\right] \mathbb{1}(0<x<\infty)\, dy$$

$$= \frac{x^{-3/2}}{\sqrt{2\pi}} e^{-\frac{1}{2x}} \sqrt{x} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} e^{-\frac{y^2}{2x}} dy}_{=1} \mathbb{1}(0<x<\infty)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2x}} \mathbb{1}(0<x<\infty).$$

So, for $0 < x < \infty$ and $-\infty < y < \infty$, we have

$$f(y|x) = \frac{\frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2+1)\right]}{\frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2x}}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} \exp\left[-\frac{y^2}{2x}\right],$$

which we recognize as the pdf of the Normal$(0, x)$ distribution, so that

$$Y \mid X = x \sim \text{Normal}(0, x).$$

(b) Find $P(Y > 1 \mid X = 4)$.

We have

$$\begin{aligned} P(Y > 1 \mid X = 4) &= P\left(\frac{Y}{2} > \tfrac{1}{2} \,\middle|\, X = 4\right) \\ &= P(Z > \tfrac{1}{2}), \quad Z \sim \text{Normal}(0,1) \\ &= 1 - \underbrace{\Phi(\tfrac{1}{2})}_{\text{pnorm}(0.5)} \\ &= 0.3085375 \end{aligned}$$

E.g. Let (U, V) have joint pdf given by

$$f(u,v) = 6(v-u)\, \mathbb{1}(0 < u < v < 1).$$

(a) Find the conditional density $f(u \mid v)$ of $U \mid V = v$.

Since

$$f(u \mid v) = \frac{f(u,v)}{f_V(v)}$$

when $f_V(v) > 0$, we must first find the marginal pdf f_V of V.

We have, for $0 < v < 1$,

$$f_V(v) = \int_{-\infty}^{\infty} 6(v-u)\, \mathbb{1}(0<u<v<\infty)\, du$$

$$= \int_0^v 6(v-u)\, du$$

$$= 6\left(vu - \frac{u^2}{2}\right)\Big|_0^v$$

$$= 6\, \frac{v^2}{2}$$

$$= 3v^2.$$

So we have, for $0<v<1$,

$$f(u|v) = \frac{6(v-u)}{3v^2}\, \mathbb{1}(0<u<v)$$

$$= 2\left(\frac{1}{v} - \frac{u}{v^2}\right) \mathbb{1}(0<u<v).$$

(b) Find $P(U < 1/4 \mid V = 1/2)$.

We have

$$P(U < \tfrac{1}{4} \mid V = 1/2) = \int_0^{1/4} 2\left(\frac{1}{(1/2)} - \frac{u}{(1/2)^2}\right) du$$

$$= 4\left(u - u^2\right)\Big|_0^{1/4}$$

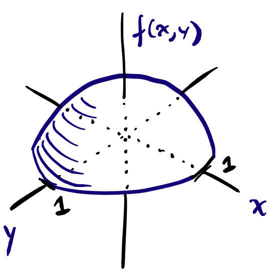
$$= 4\left[\frac{1}{4} - \left(\frac{1}{4}\right)^2\right]$$

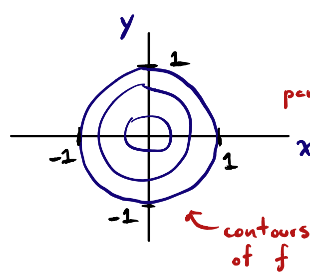
$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}.$$

E.g. (2-dimensional Epanechnikov Kernel)

Let $(X,Y) \sim f(x,y) = \frac{2}{\pi}\left[1-(x^2+y^2)\right]_+$, where $[\cdot]_+ = \max(0,\cdot)$.





f is a 2-d parabola with maximum at the origin, but equal to zero outside the unit circle.

(i) Get conditional pdf of X given that $Y=y$ for some $y \in (-1,1)$.

By the definition,

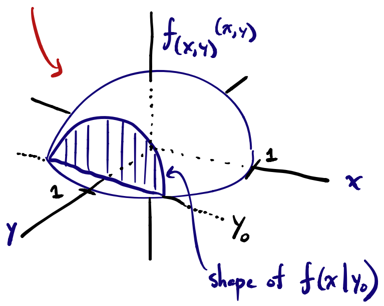
$$f(x|0) = \frac{f(x,0)}{\underbrace{f_Y(y)}_{\text{We need the marginal pdf of } Y}}.$$

First get $$f_Y(y) = \int_{\mathbb{R}} \frac{2}{\pi}\big[\underbrace{1-(x^2+y^2)}_{\text{is positive when } x^2<1-y^2 \iff |x|<\sqrt{1-y^2}}\big]_+ \, dx$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}\left[(1-y^2)-x^2\right] dx$$

$$= \frac{2}{\pi}\left[(1-y^2)x - \frac{x^3}{3}\right]\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

Slice the joint pdf at some y, then rescale so that the cross-section integrates to 1



$$= \frac{2}{\pi}\left[\left((1-y^2)\sqrt{1-y^2} - \frac{(1-y^2)^{3/2}}{3}\right) - \left((1-y^2)(-\sqrt{1-y^2}) + \frac{(1-y^2)^{3/2}}{3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{6}{3}(1-y^2)^{3/2} - \frac{2}{3}(1-y^2)^{3/2}\right]$$

$$= \frac{8}{3\pi}(1-y^2)^{3/2}.$$

Then

$$f(x|y) = \frac{\frac{2}{\pi}\left[1-(x^2+y^2)\right]_+}{\frac{8}{3\pi}(1-y^2)^{3/2}}$$

We can use an indicator function instead of $[\cdot]_+$

$$= \frac{\frac{3}{4}\left[1-(x^2+y^2)\right]}{(1-y^2)^{3/2}}\,\mathbb{1}\left(-\sqrt{1-y^2} < x < \sqrt{1-y^2}\right) \text{ for all } x\in\mathbb{R}.$$

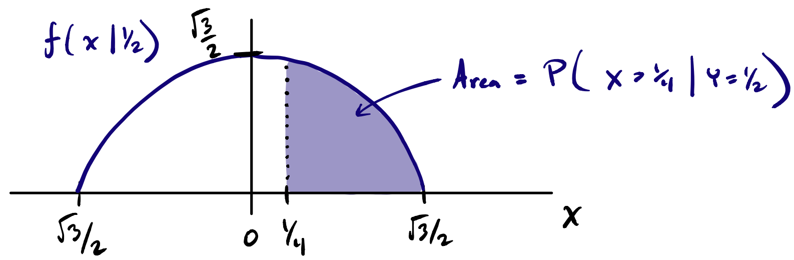
(ii) Find $P(X > 1/4 \mid Y = 1/2)$.

The conditional pdf of X given that $Y = 1/2$ is

$$f(x \mid 1/2) = \frac{\frac{3}{4}\left[1 - (x^2 + (1/2)^2)\right]}{(1 - (1/2)^2)^{3/2}} \, \mathbb{1}\left(-\sqrt{1-(\tfrac{1}{2})^2} < x < \sqrt{1-(\tfrac{1}{2})^2}\right)$$

$\left(\sqrt{1-(1/2)^2} = \sqrt{3/4} = \frac{\sqrt{3}}{2}\right)$

$$= \frac{2}{\sqrt{3}} \left(\frac{3}{4} - x^2\right) \mathbb{1}\left(-\sqrt{3}/2 < x < \sqrt{3}/2\right)$$



$$P(X > 1/4 \mid Y = 1/2) = \int_{1/4}^{\infty} \frac{2}{\sqrt{3}} \left(\frac{3}{4} - x^2\right) \mathbb{1}\left(-\sqrt{3}/2 < x < \sqrt{3}/2\right) dx$$

$\left(\frac{1}{4} < \frac{\sqrt{3}}{2}\right)$

$$= \int_{1/4}^{\sqrt{3}/2} \frac{2}{\sqrt{3}} \left(\frac{3}{4} - x^2\right) dx$$

$$= \frac{2}{\sqrt{3}} \left[\frac{3}{4}x - \frac{x^3}{3}\right]_{1/4}^{\sqrt{3}/2}$$

$$= \frac{2}{\sqrt{3}} \left[\left(\frac{3}{4}\frac{\sqrt{3}}{2} - \frac{(\sqrt{3}/2)^3}{3}\right) - \left(\frac{3}{4}\frac{1}{4} - \frac{(1/4)^3}{3}\right)\right]$$

$$= 2\left(\frac{3}{8} - \frac{1}{8}\right) - \frac{2}{\sqrt{3}}\left(\frac{3\cdot3\cdot4 - 1}{192}\right)$$

$$= \frac{1}{2} - \frac{1}{\sqrt{3}}\frac{35}{96}$$

CONDITIONAL EXPECTATION

Let (X,Y) be a pair of discrete or continuous r.v.s.

Defn: For any function $g:\mathbb{R}\to\mathbb{R}$ and any value y such that $p_Y(y)>0$ or $f_Y(y)>0$, the conditional expectation of $g(X)$ given that $Y=y$ is

$$\mathbb{E}[g(X)\mid Y=y] = \begin{cases} \sum_{x\in\mathcal{X}} g(x)\cdot p(x\mid y) & \text{if } (X,Y) \text{ is discrete} \\ \int_{\mathbb{R}} g(x)\cdot f(x\mid y)\,dx & \text{if } (X,Y) \text{ is continuous.} \end{cases}$$

And likewise for $\mathbb{E}[g(Y)\mid X=x]$.

Remark: The conditional expectation $\mathbb{E}[g(X)\mid Y=y]$ is a function of y, since X is summed / integrated out.

Set aside the function $g:\mathbb{R}\to\mathbb{R}$ for a moment:

If we write $\mathbb{E}[X\mid Y]$, without choosing a value y for Y, then $\mathbb{E}[X\mid Y]$ is a random variable, as its value depends on the value of the random variable Y.

E.g. Roll two dice. Let $X=$ sum of rolls, $Y=$ maximum of rolls.

Get conditional expectation of X given $Y=4$ and Y given $X=7$.

From before:

x	5	6	7	8
$p_{X\mid Y}(x\mid 4)$	$2/7$	$2/7$	$2/7$	$1/7$

gives $\mathbb{E}[X\mid Y=4] = \dfrac{2(5)+2(6)+2(7)+1(8)}{7}$

$= \dfrac{44}{7}$

y	4	5	6
$p_{Y\mid X}(y\mid 7)$	$2/6$	$2/6$	$2/6$

gives $\mathbb{E}[Y\mid X=7] = \dfrac{2(4)+2(5)+2(6)}{6} = 5$

E.g. Let (U,V) have joint pdf given by

$$f(u,v) = 6(v-u)\,\mathbb{1}(0<u<v<1).$$

(a) Find $\mathbb{E}[U \mid V=v]$.

We have

$$\mathbb{E}[U \mid V=v] = \int_0^v u \cdot 2\left(\frac{1}{v} - \frac{u}{v^2}\right) du$$

$$= 2\left(\frac{u^2}{2v} - \frac{u^3}{3v^2}\right)\Big|_0^v$$

$$= \frac{v}{3}.$$

(b) Give $\mathbb{E}[U \mid V=1/2]$.

We have

$$\mathbb{E}[U \mid V=1/2] = \frac{1}{2} - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

E.g. Let $(X,Y) \sim \frac{2}{\pi}\left[1-(x^2+y^2)\right]_+$.

(i) Find $\mathbb{E}[X \mid Y=y]$ for any $y \in (-1,1)$.

$$\mathbb{E}[X \mid Y=y] = \int_{\mathbb{R}} x \cdot f(x \mid y)\, dx$$

from previous work

$$= \int_{\mathbb{R}} x \cdot \frac{\frac{3}{4}\left[1-(x^2+y^2)\right]}{(1-y^2)^{3/2}} \, \mathbb{1}\left(-\sqrt{1-y^2} < x < \sqrt{1-y^2}\right) dx$$

$$= \frac{3/4}{(1-y^2)^{3/2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left[(1-y^2)x - x^3\right] dx$$

$$= \frac{3/4}{(1-y^2)^{3/2}} \left[\frac{(1-y^2)x^2}{2} - \frac{x^4}{4}\right] \Bigg|_{\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

$$= \frac{3/4}{(1-y^2)^{3/2}} \left[\left(\frac{(1-y^2)^2}{2} - \frac{(1-y^2)^2}{4}\right) - \left(\frac{(1-y^2)^2}{2} - \frac{(1-y^2)^2}{4}\right)\right]$$

$$= 0.$$

We might have guessed: A cross-section of the joint pdf at any y is symmetric about 0.

(ii) Find $\mathbb{E}\left[X^2 \mid Y=y\right]$.

$$\mathbb{E}\left[X^2 \mid Y=y\right] = \frac{3/4}{(1-y^2)^{3/2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left[(1-y^2)x^2 - x^4\right] dx$$

$$= \frac{3/4}{(1-y^2)^{3/2}} \left[(1-y^2)\frac{x^3}{3} - \frac{x^5}{5}\right] \Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

$$= \frac{3/4}{(1-y^2)^{3/2}}\left[\left(\frac{(1-y^2)^{5/2}}{3} - \frac{(1-y^2)^{5/2}}{5}\right) - \left(-\frac{(1-y^2)^{5/2}}{3} + \frac{(1-y^2)^{5/2}}{5}\right)\right]$$

$$= (1-y^2)\frac{3}{4}\left[\frac{2}{3} - \frac{2}{5}\right]$$

$$= \frac{1}{5}(1-y^2).$$

Defn: The conditional variance of X given that $Y=y$ is

$$\mathrm{Var}[X|Y=y] = \mathbb{E}\left[\left(X - \mathbb{E}[X|Y=y]\right)^2 \middle| Y=y\right].$$

Remark: If we write $\mathrm{Var}[X|Y]$, without choosing a value y for Y, then $\mathrm{Var}[X|Y]$ is a random variable, as it depends on the value of the random variable Y.

Useful expression: $\mathrm{Var}[X|Y] = \mathbb{E}[X^2|Y] - \left(\mathbb{E}[X|Y]\right)^2$

E.g. Let $(X,Y) \sim \frac{2}{\pi}\left[1-(x^2+y^2)\right]_+$.

From before: $\mathbb{E}[X|Y] = 0$ and $\mathbb{E}[X^2|Y] = \frac{1}{5}(1-Y^2)$, so

$$\mathrm{Var}[X|Y] = \frac{1}{5}(1-Y^2).$$

Then, for example, $\mathrm{Var}[X|Y=\frac{1}{2}]$ is given by

$$\mathrm{Var}[X|Y=\tfrac{1}{2}] = \frac{1}{5}\left[1-\left(\tfrac{1}{2}\right)^2\right]$$

$$= \frac{3}{20}$$

Eg. Let (X,Y) have joint pdf given by

$$f(x,y) = y(1-x)^{y-1}e^{-y}\,\mathbb{1}(0<x<1,\ 0<y<\infty).$$

(a) Find $\mathbb{E}[X \mid Y=y]$.

Use the fact that $X \mid Y=y \sim \text{Beta}(2,y)$, $\mathbb{E}[X \mid Y=y] = \dfrac{1}{1+y}$.

We can also obtain by the calculation

$$\begin{aligned}
\mathbb{E}[X \mid Y=y] &= \int_0^1 x \cdot y(1-x)^{y-1}\,dx \\
&= y \cdot \underbrace{\int_0^1 x^{2-1}(1-x)^{y-1}\,dx}_{B(2,y) = \frac{\Gamma(2)\Gamma(y)}{\Gamma(2+y)}} \\
&= y \cdot \frac{\Gamma(2)\Gamma(y)}{\Gamma(2+y)} \\
&= \frac{y\,\Gamma(y)}{(y+1)y\,\Gamma(y)} \\
&= \frac{1}{1+y}.
\end{aligned}$$

(b) Find $\text{Var}[X \mid Y=y]$.

We have, since $X \mid Y=y \sim \text{Beta}(1,y)$, that $\text{Var}[X \mid Y=y] = \dfrac{y}{(y+1)^2(y+2)}$.

We can also find this as $\mathbb{E}[X^2 \mid Y=y] - (\mathbb{E}[X \mid Y=y])^2$. We have

$$\begin{aligned}
\mathbb{E}[X^2 \mid Y=y] &= \int_0^1 x^2 \cdot y(1-x)^{y-1}\,dx \\
&= y\int_0^1 x^{3-1}(1-x)^{y-1}\,dx
\end{aligned}$$

$$= y \cdot \frac{\Gamma(3)\,\Gamma(y)}{\Gamma(3+y)}$$

$$= \frac{y \cdot 2 \cdot \Gamma(y)}{(y+2)(y+1)\,y\,\Gamma(y)}$$

$$= \frac{2}{(y+2)(y+1)}$$

so that

$$\operatorname{Var}[X \mid Y=y] = \frac{2}{(y+2)(y+1)} - \left[\frac{1}{y+1}\right]^2$$

$$= \frac{(y+1)2 - (y+2)}{(y+1)^2(y+2)}$$

$$= \frac{y}{(y+1)^2(y+2)}.$$

ADDENDUM: To find C such that $C\left[1-(x^2+y^2)\right]_+$ is a joint pdf, we do the following:

$$1 = \iint_{\mathbb{R}^2} C\left[1-(x^2+y^2)\right]_+ \,dx\,dy$$

$$= C \iint_{\mathbb{R}^2} \left[1-(x^2+y^2)\right] \mathbb{1}(x^2+y^2<1)\,dx\,dy$$

switch to polar coordinates: $r^2 = x^2+y^2$, $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$, $0<r<1$, $0<\theta<2\pi$.

$$= C\int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= C\int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 d\theta$$

$$= C\int_0^{2\pi} \frac{1}{4}\,d\theta$$

$$= C\,\frac{2\pi}{4} \quad \Rightarrow \quad C = \frac{2}{\pi}$$