STAT 511 su 2020 Lec 12 slides

Conditional distributions and conditional expectation

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These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

Conditional probability mass functions

Let (X, Y) be discrete rvs with joint pmf p and marginal pmfs p_X and p_Y.

- For any y such that $p_Y(y) > 0$, the *conditional pmf* of $X|Y = y$ is

$$p(x|y) = \frac{p(x,y)}{p_Y(y)}, \quad x \in \mathbb{R}.$$

- For any x such that $p_X(x) > 0$, the *conditional pmf* of $Y|X = x$ is

$$p(y|x) = \frac{p(x,y)}{p_X(x)}, \quad y \in \mathbb{R}.$$

Exercise: Show that the conditional pmfs are legitimate pmfs.

Let X = sum of two dice rolls, Y = max.

		$\mathcal{Y}$						
		1	2	3	4	5	6	$p_X(x)$
	2	1/36						1/36
	3		2/36					2/36
	4		1/36	2/36				3/36
	5			2/36	2/36			4/36
$\mathcal{X}$	6			1/36	2/36	2/36		5/36
	7				2/36	2/36	2/36	6/36
	8				1/36	2/36	2/36	5/36
	9					2/36	2/36	4/36
	10					1/36	2/36	3/36
	11						2/36	2/36
	12						1/36	1/36
	$p_Y(y)$	1/36	3/36	5/36	7/36	9/36	11/36	

Exercise: Tabulate $p(x|y = 4)$ and $p(y|x = 7)$.

Conditional probability density functions

Let (X, Y) be continuous rvs with joint pdf f and marginal pdfs f_X and f_Y.

- For any y such that $f_Y(y) > 0$, the *conditional pdf* of $X|Y = y$ is

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad x \in \mathbb{R}.$$

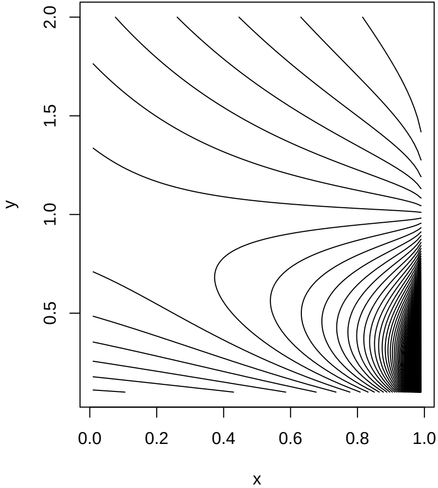
- For any x such that $f_X(x) > 0$, the *conditional pdf* of $Y|X = x$ is

$$f(y|x) = \frac{f(x,y)}{f_X(x)}, \quad y \in \mathbb{R}.$$

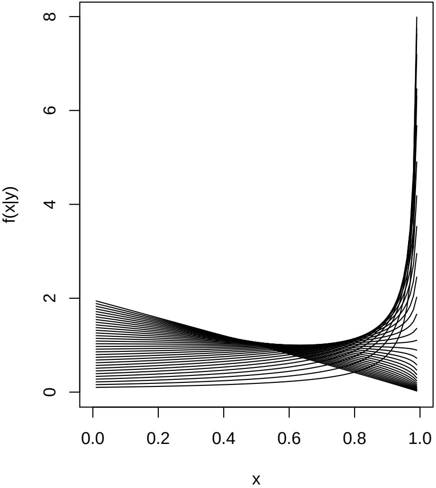
Exercise: Show that the conditional pdfs are legitimate pdfs.

Exercise: Let $(X, Y) \sim f(x, y) = y(1 - x)^{y-1} e^{-y} \mathbf{1}(0 < x < 1, 0 < y < \infty)$.

1. Find the pdf $f(x|y)$ of X given $Y = y$.
2. Find $P(X < 1/2 | Y = 2)$.



2.0
1.5
1.0
0.5
y
0.0
0.2
0.4
0.6
0.8
1.0
x

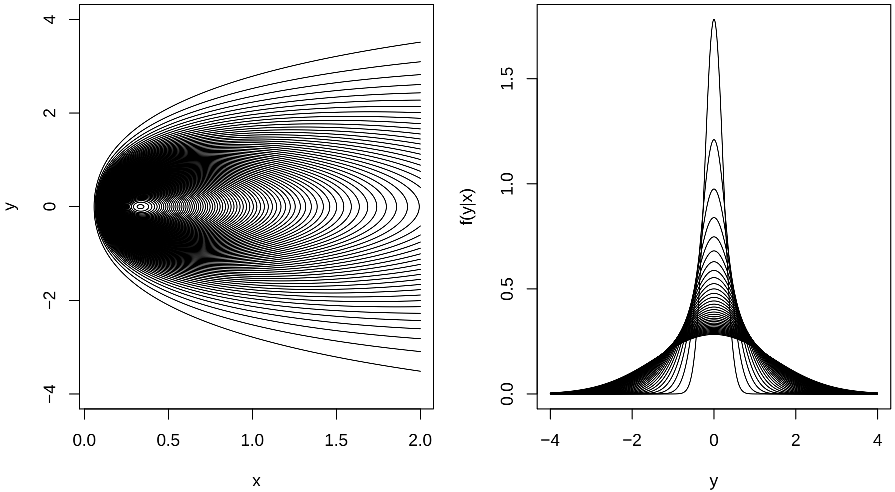


8
6
4
2
0
f(x|y)
0.0
0.2
0.4
0.6
0.8
1.0
x

Exercise: Let (X, Y) have joint pdf given by

$$f(x,y) = \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2+1)\right] \mathbf{1}(0 < x < \infty, -\infty < y < \infty).$$

1. Find the conditional pdf $f(y|x)$ of Y given $X = x$.
2. Find $P(Y > 1 | X = 4)$.



y
4
2
0
−2
−4
0.0
0.5
1.0
1.5
2.0
x
f(y|x)
0.0
0.5
1.0
1.5
−4
−2
0
2
4
y

Exercise: Let (U, V) be a pair of rvs with joint pdf given by

$$f(u, v) = 6(v - u)\mathbf{1}(0 < u < v < 1).$$

1. Find the conditional pdf $f(u|v)$ of U given $V = v$.
2. Find $P(U < 1/4|V = 1/2)$.

Conditional expectation

Let (X, Y) be discrete or continuous rvs on $\mathcal{X}$ and $\mathcal{Y}$ and $g : \mathbb{R}^2 \to \mathbb{R}$.
For any $y \in \mathcal{Y}$, the *conditional expectation* of $g(X)$ given $Y = y$ is

$$\mathbb{E}[g(X)|Y = y] = \begin{cases} \displaystyle\sum_{x \in \mathcal{X}} g(x) \cdot p(x|y) & \text{for } (X, Y) \text{ discrete} \\ \displaystyle\int_{\mathbb{R}} g(x) \cdot f(x|y) & \text{for } (X, Y) \text{ continuous} \end{cases}$$

The conditional expectation $\mathbb{E}[g(Y)|X = x]$ is likewise defined.

Note that $\mathbb{E}[g(X)|Y = y]$ is a function of y, since X is summed/integrated out.

We often have $g(x) = x$, so that we consider $\mathbb{E}[X|Y = y]$.

If we write $\mathbb{E}[X|Y]$, without specifying a value y for Y, then $\mathbb{E}[X|Y]$ is a rv.

Exercise: Let (U, V) be a pair of rvs with joint pdf given by

$$f(u, v) = 6(v - u)\mathbf{1}(0 < u < v < 1).$$

1. Find $\mathbb{E}[U|V = v]$.
2. Find $\mathbb{E}[U|V = 1/2]$.

Conditional variance

The conditional variance of X given that $Y = y$ is

$$\text{Var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y].$$

If we write $\text{Var}[X|Y]$, without specifying a value y for Y, then $\text{Var}[X|Y]$ is a rv.

Useful expression: $\text{Var}[X|Y] = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$

Exercise: Let $(X, Y) \sim f(x, y) = y(1-x)^{y-1}e^{-y}\mathbf{1}(0 < x < 1, 0 < y < \infty)$.

1. Find $\mathbb{E}[X|Y = y]$.
2. Find $\text{Var}[X|Y = y]$.
3. Find $\text{Var}[X|Y = 2]$.

Exercise: Let (X, Y) have joint pdf given by

$$f(x, y) = \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2 + 1)\right] \mathbf{1}(0 < x < \infty, -\infty < y < \infty).$$

1. Find $\mathbb{E}[Y|X = x]$.
2. Find $\text{Var}[Y|X = x]$.
3. Find $\text{Var}[Y|Y = 2]$.