INDEPENDENCE OF RANDOM VARIABLES

Recall: Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

Defn: Let (X,Y) be a pair of random variables where X has support $\mathcal{X}$ and Y has support $\mathcal{Y}$. Then X and Y are called independent random variables if for any sets $A \in \underbrace{\mathcal{E}_{\mathcal{X}}}$ and $B \in \underbrace{\mathcal{E}_{\mathcal{Y}}}$

Special collections of sets of interest in $\mathcal{X}$ and $\mathcal{Y}$

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

Basically, we consider all possible events concerning X and all possible events concerning Y. If every pair of X-and-Y events is independent, we call the r.v.s X and Y independent r.v.s.

Theorem: If (X,Y) is a pair of discrete r.v.s with joint pmf p and marginal pmfs p_X and p_Y, then X and Y are independent iff

$$p(x,y) = p_X(x)\,p_Y(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

If (X,Y) is a pair of continuous r.v.s with joint pdf f and marginal pdfs f_X and f_Y, then X and Y are independent iff

$$f(x,y) = f_X(x)\,f_Y(y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

* It is equivalent to say that X and Y are independent iff

$$f(x|y) = f_X(x) \quad \text{for all } x \in \mathbb{R}, \text{ all } y \in \{y : f_Y(y) > 0\},$$

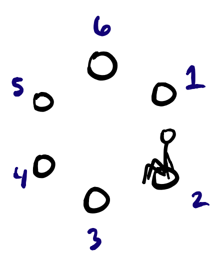
or

$$p(x|y) = p_X(x) \quad \text{for all } x \in \mathbb{R}, \text{ all } y \in \{y : p_Y(y) > 0\},$$

(Independence of X and Y means that the distribution of X is unaffected by the value of Y and vice versa.)

that is if the conditional pdfs/pmfs are equal to the marginal pdfs/pmfs.

E.g. There are 6 chairs in a circle numbered $1, \ldots, 6$.



- Let X be the roll of a die and sit in chair X
- Roll die again and move that many chairs clockwise
- Let Y be the number of the chair in which you now sit

Are X and Y independent?

• Tabulate the joint distribution of (X, Y).

Y \ X	1	2	3	4	5	6
1	$1/36$					
2						
3						
4					$1/36$	
5						
6						

To begin in chair 1 and end in chair 1, we must roll $(1,6)$, which we do with probability $1/36$

Must roll $(5,5)$, which we do with probability $1/36$

We find that $P\big((X,Y)=(x,y)\big) = \frac{1}{36}$ for all $(x,y) \in \{1,\dots,6\}\times\{1,\dots,6\}$

• Get marginals: $P_X(X=x) = 1/6$ for $x \in \{1,\dots,6\}$

$P_Y(Y=y) = 1/6$ for $y \in \{1,\dots,6\}$

• Check: $P\big((X,Y)=(x,y)\big) = P_X(X=x)\,P_Y(Y=y)$ for all (x,y),

So yes, X and Y are independent, meaning that the chair you end on is not affected by the chair you started on.

E.g. Suppose X and Y are independent r.v.s with marginal pmfs

$$p_X(x) = p^x (1-p)^{1-x}\, \mathbb{1}(x \in \{0,1\})$$

$$p_Y(y) = \binom{3}{y} q^y (1-q)^{3-y}\, \mathbb{1}(y \in \{0,1,2,3\})$$

Then the joint pmf of X and Y is the product of the marginals:

$$p(x,y) = p^x(1-p)^x \binom{3}{y} q^y (1-q)^{3-y}\, \underbrace{\mathbb{1}(x \in \{0,1\},\ y \in \{0,1,2,3\})}_{=\mathbb{1}(x\in\{0,1\})\cdot\mathbb{1}(y\in\{0,1,2,3\})}$$

E.g. Let (X,Y) be a continuous pair of r.v.s with joint pdf given by

$$f(x,y) = \begin{cases} \frac{6}{5}\left[1-(x-y)^2\right] & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent?

- Find marginal pdfs:

$$f_X(x) = \int_{\mathbb{R}} \frac{6}{5}\left[1-(x-y)^2\right] \mathbb{1}(0 \le x \le 1, 0 \le y \le 1)\, dy$$

(See lec 11) $\vdots$

$$= \frac{2}{5}\left(2+3x-3x^2\right) \mathbb{1}(0 \le x \le 1)$$

We get the same for Y:

$$f_Y(y) = \frac{2}{5}\left(2+3y-3y^2\right) \mathbb{1}(0 \le y \le 1).$$

Now $$f_X(x) f_Y(y) = \frac{4}{25}\left(2+3x-3x^2\right)\left(2+3y-3y^2\right) \mathbb{1}(0 \le x \le 1, 0 \le y \le 1).$$

This is not equal to $f(x,y) = \frac{6}{5}\left[1-(x-y)^2\right] \mathbb{1}(0 \le x \le 1, 0 \le y \le 1)$, for example, for $(x,y) = (1/2, 1/2)$.

It may be a hassle to compute the marginals f_X, f_Y or p_X, p_Y. There is an easier way to check independence:

Theorem: Let (X,Y) be a pair of either discrete or continuous r.v.s with joint pmf p or joint pdf f. Then X and Y are independent iff there exists functions g and h such that

$$p(x,y) = g(x)h(y) \quad \text{for all } (x,y) \in \mathbb{R}^2$$

or

$$f(x,y) = g(x)h(y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

* So instead of finding the marginal pmfs or pdfs, we must only find the functions g and h in order to establish independence of X and Y.

E.g. Let (X,Y) have the joint pdf

$$f(x,y) = 48xy(y-xy)\mathbb{1}(0 \le x \le 1,\ 0 \le y \le 1).$$

We can write $f(x,y) = 48x(1-x)^2\mathbb{1}(0 \le x \le 1) \cdot y^3\mathbb{1}(0 \le y \le 1)$,

so putting $g(x) = 48x(1-x)^2\mathbb{1}(0 \le x \le 1)$ and $h(y) = y^3\mathbb{1}(0 \le y \le 1)$,

$$f(x,y) = g(x)h(y).$$

These do not have to be pdfs, i.e. they need not integrate to 1

Therefore X and Y are independent.

E.g. Let (X,Y) have the joint pdf

$$f(x,y) = \frac{1}{4\pi}\exp\left[-\frac{x^2+y^2-2x+1}{4}\right].$$

Then $f(x,y) = g(x)\cdot h(y)$ for

$$g(x) = \frac{1}{4\pi}\exp\left[-\frac{x^2-2x+1}{4}\right], \qquad h(y) = \exp\left[-\frac{y^2}{4}\right],$$

so X and Y are independent.

E.g. Let (X,Y) have the joint pdf

$$f(x,y) = \frac{6}{5}\left[1-(x-y)^2\right]\mathbb{1}(0 \le x \le 1,\ 0 \le y \le 1).$$

We cannot find $g(x)$ and $h(y)$ such that $f(x,y) = g(x)\cdot h(y)$, so X and Y are not independent.

The next theorem says that if two r.v.s are independent, then the expected value of their product is equal to the product of their expected values. This comes from the fact that their joint pdf/pmf, when they are independent, is equal to the product of their marginal pdfs/pmfs.

Theorem: Let X and Y be independent random variables. Then

$$\mathbb{E}XY = \mathbb{E}X\,\mathbb{E}Y.$$

Moreover, for any functions $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$,

$$\mathbb{E}g(X)h(Y) = \mathbb{E}g(X)\,\mathbb{E}h(Y).$$

Proof: Let $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$. Then if X and Y are continuous with joint pdf $f_{(X,Y)}$,

$$\mathbb{E}\,g(X)h(Y) = \iint_{\mathbb{R}^2} g(x)g(y)\, f_{(X,Y)}(x,y)\,dx\,dy$$

(If X and Y are discrete, do the same steps, replacing pdfs with pmfs and integrals with sums.)

$$= \iint_{\mathbb{R}^2} g(x) f_X(x)\cdot h(y) f_Y(y)\cdot dx\,dy$$

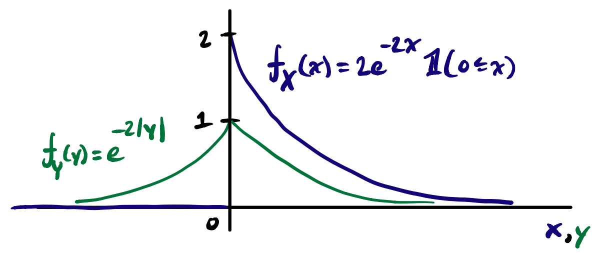
$$= \int_{\mathbb{R}} g(x) f_X(x)\,dx \cdot \int_{\mathbb{R}} h(y) f_Y(y)\,dy$$

$$= \mathbb{E}g(X)\,\mathbb{E}h(Y).$$

E.g. Let X and Y be independent random variables with marginal pdfs

$$f_X(x) = 2e^{-2x}\,\mathbb{1}(x\geq 0)$$

$$f_Y(y) = e^{-2|y|}\,\mathbb{1}(-\infty < y < \infty)$$



Then $\mathbb{E}XY^2 = \mathbb{E}X\,\mathbb{E}Y^2$, where $\mathbb{E}X = \frac{1}{2}$, since $X \sim \text{Exponential}(\lambda = \frac{1}{2})$

and

$$\mathbb{E}Y^2 = \int_{-\infty}^{\infty} y^2 e^{-2|y|}\,dy$$

$$|y| = \begin{cases} y & \text{if } y>0 \\ 0 & \text{if } y=0 \\ -y & \text{if } y<0 \end{cases}$$

$$= \int_{-\infty}^{0} y^2 e^{2y}\,dy + \int_0^{\infty} y^2 e^{-2y}\,dy$$

$$\int_{-\infty}^{0} y^2 e^{2y}\,dy = -\int_{\infty}^{0} w^2 e^{-2w}\,dw = \int_0^{\infty} w^2 e^{-2w}\,dw$$

$w=-y$, $y=-w$, $dy = (-1)dw$

$$= 2\int_0^{\infty} y^2 e^{-2y}\,dy$$

$u = 2y$, $y = \frac{1}{2}u$

$dy = \frac{1}{2}du$

$$= \frac{1}{4}\int_0^{\infty} u^2 e^{-u}\,du$$

$$= \Gamma(3)/4$$

$$= \frac{1}{2}$$

So $\quad \mathbb{E} X Y^2 = \mathbb{E} X \, \mathbb{E} Y^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$

The following result helps us find the distribution of sums of independent random variables. It says that the mgf of the sum of two independent r.v.s is equal to the product of the two mgfs.

Theorem: Let X and Y be independent random variables with mgfs M_X and M_Y, respectively. Then the mgf of the random variable $V = X + Y$ is given by

$$M_V(t) = M_X(t) M_Y(t).$$

Proof: $M_V(t) = \mathbb{E}\, e^{tV} = \mathbb{E}\, e^{t(X+Y)} = \mathbb{E}\, e^{tX} e^{tY} = \mathbb{E}\, e^{tX} \, \mathbb{E}\, e^{tY} = M_X(t) M_Y(t).$

by independence of X and Y

E.g. Let X and Y be independent r.v.s such that $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$ and let $U = X + Y$.

Then $U = \#$ successes in $n+m$ Bernoulli trials with success probability p, so $U \sim \text{Binomial}(n+m, p)$.

Indeed, the mgf of U is given by

$$M_U(t) = M_X(t) M_Y(t) = [pe^t + (1-p)]^n \, [pe^t + (1-p)]^m$$
$$= [pe^t + (1-p)]^{n+m},$$

which is the mgf of the $\text{Binomial}(n+m, p)$ distribution.

E.g. (Sum of independent Normal r.v.s): Let X and Y be independent r.v.s such that

$$X \sim \text{Normal}(\mu_X, \sigma_X^2) \quad \text{and} \quad Y \sim \text{Normal}(\mu_Y, \sigma_Y^2).$$

(marginal distribution of X) (marginal distribution of Y)

Then $U = X + Y$ has the $\text{Normal}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$ distribution, since

$$M_U(t) = M_X(t)\,M_Y(t) = e^{\mu_X t + \sigma_X^2 t^2/2}\; e^{\mu_Y t + \sigma_Y^2 t^2/2}$$

$$= e^{(\mu_X+\mu_Y)t \;+\; (\sigma_X^2+\sigma_Y^2)t^2/2},$$

which is the mgf of the Normal$(\mu_X+\mu_Y,\ \sigma_X^2+\sigma_Y^2)$ distribution.

E.g. (Sum of independent Poissons) Let X_1 and X_2 be independent r.v.s such that

$$X_1 \sim \text{Poisson}(\lambda_1) \quad \text{and} \quad X_2 \sim \text{Poisson}(\lambda_2).$$

Then $Y = X_1 + X_2 \sim \text{Poisson}(\lambda_1+\lambda_2)$, since

$$M_Y(t) = M_{X_1}(t)\,M_{X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)},$$

which is the mgf of the Poisson$(\lambda_1+\lambda_2)$ distribution.