STAT 511 su 2020 Lec 13 slides

Independence of random variables

Karl B. Gregory

University of South Carolina

These slides are an instructional aid; their sole purpose is to display, during the lecture, definitions, plots, results, etc. which take too much time to write by hand on the blackboard. They are not intended to explain or expound on any material.

- Have defined independence of two events $A, B \subset \mathcal{S}$ as the property that

$$P(A \cap B) = P(A)P(B)$$

- Now we want a notion of non-linked-ness for two random variables.

- For a pair of rvs (X, Y), when should we call X and Y "independent"?

Let (X, Y) be rvs on $\mathcal{X}$ and $\mathcal{Y}$, and $\mathcal{E}_{\mathcal{X}}$ and $\mathcal{E}_{\mathcal{Y}}$ be collections of sets in $\mathcal{X}$ and $\mathcal{Y}$.

Independence of random variables

Then X and Y are *independent* if for any sets $A \in \mathcal{E}_{\mathcal{X}}$ and $B \in \mathcal{E}_{\mathcal{Y}}$

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

Consider all possible pairs of events concerning X and Y; if all pairs independent, we call X and Y independent.

Theorem (Independence iff joint is product of marginals)

- *Let (X, Y) be discrete rvs with joint pmf p and marginal pmfs p_X and p_Y.*

  Then X and Y are independent iff

  $$p(x, y) = p_X(x)p_Y(y) \text{ for all } (x, y) \in \mathbb{R}^2.$$

- *Let (X, Y) be continuous rvs with joint pdf f and marginal pdfs f_X and f_Y.*

  Then X and Y are independent iff

  $$f(x, y) = f_X(x)f_Y(y) \text{ for all } (x, y) \in \mathbb{R}^2.$$

Check independence by checking whether the joint is the product of the marginals.

Exercise: Suppose there are six chairs in a circle numbered $1, \ldots, 6$. Then:

- Let X be the roll of a die and sit in chair X.
- Roll die again and move that many chairs clockwise.
- Let Y be the number of the chair in which you now sit.

Are X and Y independent?

Exercise: Let X and Y be independent rvs with marginal pmfs given by

$$p_x(x) = p^x(1-p)^{1-x} \cdot \mathbf{1}(x \in \{0,1\})$$
$$p_Y(y) = \binom{3}{y}\eta^y(1-\eta)^{3-y} \cdot \mathbf{1}(y \in \{0,1,2,3\})$$

Give the joint pmf of the rv pair (X, Y).

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = \frac{6}{5}[1 - (x - y)^2] \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

Exercise: Let X_1 and X_2 be independent rvs with the Exponential(1) distribution.

1. Give the joint pdf of (X_1, X_2)
2. Find $P(X_2 < X_1 < 2X_2)$.

Theorem (Easier way to check independence of two rvs)

Let (X, Y) be discrete or continuous rvs with joint pmf p or joint pdf f.

Then X and Y are independent iff there exist functions g and h such that

- $p(x, y) = g(x)h(y)$ *for all* $(x, y) \in \mathbb{R}^2$ *for* (X, Y) *discrete.*
- $f(x, y) = g(x)h(y)$ *for all* $(x, y) \in \mathbb{R}^2$ *for* (X, Y) *continuous.*

Check if the joint is the product of a function of just x and a function of just y.

Exercise: Let (X, Y) have the joint pdf

$$f(x, y) = 48xy(y - xy) \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = \frac{6}{5}[1 - (x - y)^2] \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = (x + y) \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

Exercise: Let (X, Y) have the joint pdf

$$f(x,y) = \frac{1}{4\pi} \exp\left[-\frac{x^2 + y^2 - 2x + 1}{4}\right] \quad \text{for all } x, y \in \mathbb{R}.$$

Check whether X and Y are independent.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = y(1 - x)^{y-1}e^{-y}\mathbf{1}(0 < x < 1, 0 < y < \infty)$$

Check whether X and Y are independent.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x, y) = \frac{1}{2\pi} x^{-3/2} \exp\left[-\frac{1}{2x}(y^2 + 1)\right] \mathbf{1}(0 < x < \infty, -\infty < y < \infty)$$

Check whether X and Y are independent.

Exercise: Let (X, Y) be a pair of rvs with joint pdf given by

$$f(x,y) = 4xy \cdot \mathbf{1}(0 < x < 1, 0 < y < 1).$$

Check whether X and Y are independent.

Exercise: Let (U, V) be a pair of rvs with joint pdf given by

$$f(u, v) = 6(v - u)\mathbf{1}(0 < u < v < 1).$$

Check whether U and V are independent.

Theorem (Expectation of the product of independent rvs)

Let X and Y be independent rvs. Then

$$\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y.$$

Moreover, for any functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}g(X)h(Y) = \mathbb{E}g(X)\mathbb{E}h(Y).$$

Exercise: Prove the result.

Exercise: Let X and Y be independent rvs with marginal pdfs

$$f_X(x) = 2e^{-2x}\mathbf{1}(x > 0)$$
$$f_Y(y) = e^{-2|y|}\mathbf{1}(-\infty < y < \infty)$$

Find $\mathbb{E}XY^2$.

Exercise: Let X and Y be independent rvs such that

$$X \sim \text{Beta}(3,4)$$
$$Y \sim \text{Beta}(1,2)$$

Find $\mathbb{E}[XY]$.

Theorem (mgf of a sum of independent random variables)

Let X *and* Y *be independent rvs with mgfs* M_X *and* M_Y.

Then the mgf of $V = X + Y$ *is*

$$M_V(t) = M_X(t) M_Y(t).$$

Exercise: Prove the result.

Exercise: Let X and Y be independent rvs such that

$$\begin{aligned} X &\sim \text{Binomial}(n, p) \\ Y &\sim \text{Binomial}(m, p) \end{aligned}$$

Find the distribution of $U = X + Y$.

Exercise: Let X and Y be independent rvs such that

$$X \sim \text{Normal}(\mu_X, \sigma_X^2)$$
$$Y \sim \text{Normal}(\mu_Y, \sigma_Y^2)$$

Find the distribution of $U = X + Y$.

Exercise: Let X_1 and X_2 be independent rvs such that

$$X_1 \sim \text{Poisson}(\lambda_1)$$
$$X_2 \sim \text{Poisson}(\lambda_2)$$

Find the distribution of $Y = X_1 + X_2$.