COVARIANCE AND CORRELATION

Let (X,Y) be a pair of r.v.s, discrete or continuous, with means and variances

$$\mathbb{E}X = \mu_X \,, \quad \mathrm{Var}\, X = \sigma_X^2$$
$$\mathbb{E}Y = \mu_Y \,, \quad \mathrm{Var}\, Y = \sigma_Y^2 \,.$$

The following two quantities describe the strength of linear relationship between X and Y.

Defn: The **covariance** between X and Y is

$$\mathrm{Cov}(X,Y) = \mathbb{E}\,(X-\mu_X)(Y-\mu_Y) =: \sigma_{XY}$$

We sometimes use σ_{XY} to denote $\mathrm{Cov}(X,Y)$.

Defn: The **correlation** between X and Y is

$$\mathrm{corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\, X}\,\sqrt{\mathrm{Var}\, Y}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} =: \rho_{XY}$$

we sometimes use ρ_{XY} to denote $\mathrm{corr}(X,Y)$

Remark: The correlation ρ_{XY} is a unitless measure of the strength of linear relationship between X and Y; the division of $\mathrm{Cov}(X,Y)$ by $\sigma_X \sigma_Y$ results in a quantity bounded between -1 and 1. That is, we always have

$$-1 \le \rho_{XY} \le 1 \,.$$

A proof of this will be given in the addendum to these notes.

Useful expression: $\mathrm{Cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y$.

Result: For $a,b,c,d \in \mathbb{R}$, we have

$$\mathrm{Cov}(aX+b,\, cY+d) = ab\,\mathrm{Cov}(X,Y)$$
$$\mathrm{corr}(aX+b,\, cY+d) = \mathrm{corr}(X,Y).$$

Ex: Let (X,Y) have joint pdf given by

$$f(x,y) = \frac{1}{8}(x+y)\,\mathbb{1}(0<x<2,\ 0<y<2).$$

(a) Find $\mathrm{Cov}(X,Y)$.

We have

$$\mathbb{E}XY = \int_0^2 \int_0^2 xy \cdot \frac{1}{8}(x+y)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \int_0^2 (x^2y + xy^2)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left(\frac{x^3}{3}y + \frac{x^2}{2}y^2\right)\Big|_0^2 dy$$

$$= \frac{1}{8}\int_0^2 \frac{8}{3}y + \frac{4}{2}y^2\ dy$$

$$= \frac{1}{8}\left(\frac{8}{3}\frac{y^2}{2} + \frac{4}{2}\frac{y^3}{3}\right)\Big|_0^2$$

$$= \frac{1}{6}\cdot 4 + \frac{4}{6}$$

$$= \frac{4}{3}.$$

To find $\mathbb{E}X$ and $\mathbb{E}Y$, we need the marginal pdfs f_X and f_Y.

For $x \in (0,2)$, we have

$$f_X(x) = \int_0^2 \frac{1}{8}(x+y)\,dy = \frac{1}{8}\left(xy + \frac{y^2}{2}\right)\Big|_0^2 = \frac{1}{8}(2x+2) = \frac{1}{4}(x+1).$$

So

$$\mathbb{E}X = \int_0^2 x \cdot \frac{1}{4}(x+1)\,dx = \frac{1}{4}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^2 = \frac{1}{4}\left(\frac{8}{3} + \frac{4}{2}\right) = \frac{7}{6}.$$

We also get $\mathbb{E}Y = 7/6$.

So

$$\mathrm{Cov}(X,Y) = \frac{4}{3} - \frac{7}{6}\cdot\frac{7}{6} = \frac{48-49}{36} = -\frac{1}{36}.$$

(b) Find $\mathrm{corr}(X,Y)$.

We have $\mathrm{Var}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2$, with

$$\mathbb{E}X^2 = \int_0^2 x^2 \cdot \frac{1}{4}(x+1)\,dx = \frac{1}{4}\left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_0^2 = \frac{1}{4}\left(\frac{16}{4} + \frac{8}{3}\right) = \frac{5}{3},$$

so that

$$\mathrm{Var}\,X = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{60-49}{36} = \frac{11}{36}.$$

We also get $\mathrm{Var}\,Y = \frac{11}{36}$.

So

$$\mathrm{corr}(X,Y) = \frac{-1/36}{\sqrt{\frac{11}{36}\cdot\frac{11}{36}}} = -\frac{1}{11}.$$

E.g. Roll two dice and let X be the maximum and Y be the minimum of the rolls.

(i) Find $\mathrm{Cov}(X, Y)$.

The joint pmf p of X and Y takes the values:

y = min \ x = max	1	2	3	4	5	6	$P_Y(Y=y)$
1	$1/36$	$2/36$	$2/36$	$2/36$	$2/36$	$2/36$	$11/36$
2		$1/36$	$2/36$	$2/36$	$2/36$	$2/36$	$9/36$
3			$1/36$	$2/36$	$2/36$	$2/36$	$7/36$
4				$1/36$	$2/36$	$2/36$	$5/36$
5					$1/36$	$2/36$	$3/36$
6						$1/36$	$1/36$
$P_X(X=x)$	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$	

$$\mathbb{E}X = \frac{1(1) + 2(3) + 3(5) + 4(7) + 5(9) + 6(11)}{36} = \frac{161}{36}$$

$$\mathbb{E}Y = \frac{1(11) + 2(9) + 3(7) + 4(5) + 5(3) + 6(1)}{36} = \frac{91}{36}$$

$$\mathbb{E}XY = \frac{1}{36}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right]$$

$$+ \frac{2}{36}\left[1(2+3+4+5+6) + 2(3+4+5+6) + 3(4+5+6) + 4(5+6) + 5(6)\right]$$

$$= \frac{1}{36}[91] + \frac{2}{36}[20 + 36 + 45 + 44 + 30]$$

$$= \frac{441}{36}$$

So $\mathrm{Cov}(X,Y) = \frac{441}{36} - \left(\frac{91}{36}\right)\left(\frac{161}{36}\right)$

$= \frac{1225}{1296} \approx .945$

(ii) Find $\mathrm{corr}(X,Y)$.

$$\mathbb{E}X^2 = \frac{1^2(2) + 2^2(3) + 3^2(5) + 4^2(7) + 5^2(9) + 6^2(11)}{36} = \frac{791}{36}$$

So $\mathrm{Var}\ X = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = \frac{2555}{1296}$

$$\mathbb{E}Y^2 = \frac{1^2(11) + 2^2(9) + 3^2(7) + 4^2(5) + 5^2(3) + 6^2(1)}{36} = \frac{301}{36}$$

So $\mathrm{Var}\ Y = \frac{301}{36} - \left(\frac{91}{36}\right)^2 = \frac{2555}{1296}$.

Then $\mathrm{corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,X}\sqrt{\mathrm{Var}\,Y}} = \frac{\left(1225/1296\right)}{\sqrt{\frac{2555}{1296}}\sqrt{\frac{2555}{1296}}} = \frac{1225}{2555} = \frac{245}{511} \approx .479$

Thm: If X and Y are independent, then $\mathrm{Cov}(X,Y) = 0$.

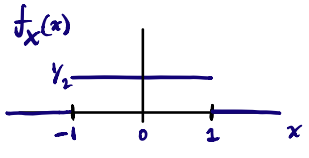
Proof: $\mathrm{Cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y = \mathbb{E}X\,\mathbb{E}Y - \mathbb{E}X\,\mathbb{E}Y = 0$.

$\mathbb{E}XY = \mathbb{E}X\,\mathbb{E}Y$, because X and Y are independent.

Remark: If $\mathrm{Cov}(X,Y) = 0$, X and Y are not necessarily independent!

E.g.: Let $X \sim \mathrm{Uniform}(-1,1)$ and let $Y = X^2$ (X and Y are not independent).

$$\mathrm{Cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y$$
$$= \mathbb{E}X^3 - \mathbb{E}X\,\mathbb{E}X^2$$
$$= 0,$$



since $\mathbb{E}X^3 = \int_{-1}^{1} x^3 \cdot \frac{1}{2}\,dx = \frac{x^4}{8}\Big|_{-1}^{1} = 0$ and $\mathbb{E}X = \int_{-1}^{1} x \cdot \frac{1}{2}\,dx = 0$.

So $\mathrm{Cov}(X,Y) = 0$, but X and Y are not independent.

Ex: Let (X,Y) have joint pdf

$$f(x,y) = \frac{1}{2}\frac{1}{|x|} e^{-y/|x|} \mathbb{1}\left(x \in (-1,1)\setminus\{0\},\ 0 < y\right).$$

(a) Check whether X and Y are independent.

We cannot find a factorization of $f(x,y)$ such that

$$f(x,y) = g(x)h(y),$$

so X and Y are not independent.

(b) Compute $\mathrm{Cov}(X,Y)$.

$$\mathbb{E}XY = \int_{-1}^{1}\int_{0}^{\infty} x\cdot y \,\frac{1}{2}\frac{1}{|x|} e^{-y/|x|}\, dy\, dx$$

$$= \int_{-1}^{1} \frac{x}{2} \underbrace{\int_{0}^{\infty} \frac{y}{|x|} e^{-y/|x|}\, dy}_{\text{Expected value of Exponential}(|x|)}\, dx$$

$$= \int_{-1}^{1} \frac{x}{2}|x|\, dx$$

$$= \int_{-1}^{0} -\frac{x^2}{2}\, dx + \int_{0}^{1} \frac{x^2}{2}\, dx$$

$$= -\frac{x^3}{6}\Big|_{-1}^{0} + \frac{x^3}{6}\Big|_{0}^{1}$$

$$= -\frac{1}{6} + \frac{1}{6}$$

$= 0.$

To find $\mathbb{E}X$, we need the marginal pdf f_X of X.

$$f_X(x) = \int_0^\infty \frac{1}{2} \frac{1}{|x|} e^{-y/|x|} \, dy \quad \mathbb{1}(x \in (-1,1) \setminus \{0\})$$

$$= \frac{1}{2} \mathbb{1}(x \in (-1,-1) \setminus \{0\}).$$

So

$$\mathbb{E}X = \int_{-\infty}^{\infty} \frac{1}{2} \mathbb{1}(x \in (-1,-1) \setminus \{0\}) \, dx$$

single point makes no contribution to the integral, so we can just integrate from -1 to 1.

$$= \int_{-1}^{1} \frac{1}{2} \, dx$$

$$= \frac{x}{2} \Big|_{-1}^{1}$$

$$= 0.$$

Therefore, $\mathrm{Cov}(X,Y) = \underbrace{\mathbb{E}XY}_{=0} - \underbrace{\mathbb{E}X}_{=0} \mathbb{E}Y = 0.$

doesn't matter

The covariance comes into play when we wish to compute the variance of a sum of r.v.s.

Thm: For any $a, b \in \mathbb{R}$,

$$\operatorname{Var}(aX + bY) = a^2 \operatorname{Var} X + b^2 \operatorname{Var} Y + 2ab \operatorname{Cov}(X, Y).$$

Proof:

$$\begin{aligned}
\operatorname{Var}(aX + bY) &= \mathbb{E}\left[aX + bY - \mathbb{E}(aX + bY) \right]^2 \\
&= \mathbb{E}\left[a(X - \mathbb{E}X) + b(Y - \mathbb{E}Y) \right]^2 \\
&= \mathbb{E}\left[a^2 (X - \mathbb{E}X)^2 + b^2 (Y - \mathbb{E}Y)^2 + 2ab (X - \mathbb{E}X)(Y - \mathbb{E}Y) \right] \\
&= a^2 \underbrace{\mathbb{E}(X - \mathbb{E}X)^2}_{\operatorname{Var} X} + b^2 \underbrace{\mathbb{E}(Y - \mathbb{E}Y)^2}_{\operatorname{Var} Y} + 2ab \underbrace{\mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y)}_{\operatorname{Cov}(X, Y)}
\end{aligned}$$

E.g. Suppose $\operatorname{Var} X = 2$, $\operatorname{Var} Y = 3$, and $\operatorname{Cov}(X, Y) = -3/2$.

Find $\operatorname{Var}(3X - Y)$. ($a = 3$, $b = -1$)

$$\begin{aligned}
\operatorname{Var}(3X - Y) &= (3)^2 \operatorname{Var} X + (-1)^2 \operatorname{Var} Y + 2(3)(-1) \operatorname{Cov}(X, Y) \\
&= 9(2) + 1(3) + 2(3)(-1)(-3/2) \\
&= 30
\end{aligned}$$

We can extend the previous theorem to a sum of any number n of r.v.s.

Thm: Let $X_1, \ldots, X_n$ be r.v.s and let $a_1, \ldots, a_n \in \mathbb{R}$. Then

$$\operatorname{Var}\left(\sum_{i=1}^n a_i X_i \right) = \sum_{i=1}^n a_i^2 \operatorname{Var}(X_i) + 2 \sum_{i>j} a_i a_j \operatorname{Cov}(X_i, X_j).$$

Proof:

$$\begin{aligned}
\operatorname{Var}\left(\textstyle\sum a_i X_i \right) &= \mathbb{E}\left[\sum_{i=1}^n a_i X_i - \mathbb{E}\left(\sum_{i=1}^n a_i X_i \right) \right]^2 \\
&= \mathbb{E}\left[\sum_{i=1}^n a_i (X_i - \mathbb{E}X_i) \right]^2 \\
&= \mathbb{E}\left[\sum_{i=1}^n a_i^2 (X_i - \mathbb{E}X_i)^2 + 2 \sum_{i>j} a_i a_j (X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j) \right]
\end{aligned}$$

$$\begin{aligned}
(c_1 + c_2 + \cdots + c_n)^2 &= c_1^2 + c_1 c_2 + \cdots + c_1 c_n \\
&\quad c_2 c_1 + c_2^2 + \cdots + c_2 c_n \\
&\quad \vdots \quad\quad \vdots \quad\quad\quad \vdots \\
&\quad c_n c_1 + c_n c_2 + \cdots + c_n^2 \\
&= \sum_{i=1}^n c_i^2 + 2 \sum_{i>j} c_i c_j
\end{aligned}$$

$$= \sum_{i=1}^{n} a_i^2 \, \mathbb{E}(X_i - \mathbb{E}X_i)^2 + 2 \sum_{i>j} a_i a_j \, \mathbb{E}(X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)$$

$$= \sum_{i=1}^{n} a_i^2 \, \mathrm{Var}(X_i) + 2 \sum_{i>j} a_i a_j \, \mathrm{Cov}(X_i, X_j).$$

Corollary: Let $X_1, \ldots, X_n$ be independent r.v.s with common variance σ^2 and let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

Proof: $\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)$

$$= \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \mathrm{Var}(X_i) + 2 \sum_{i>j} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \mathrm{Cov}(X_i, X_j)$$

$= 0$, since X_i, X_j independent for all i, j

$$= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \sigma^2$$

$$= \frac{\sigma^2}{n}.$$

We now introduce the bivariate Normal distribution, in which the correlation between the variables appears as a parameter.

BIVARIATE NORMAL DISTRIBUTION

The random variable pair (X, Y) has the bivariate Normal distribution if the joint pdf of (X, Y) is

$$f(x, y; \mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho) = \frac{1}{2\pi} \frac{1}{\sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho \left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right] \right),$$

where

- $\mu_X \in \mathbb{R}$ and $\sigma_X^2 \in (0, \infty)$ are the marginal mean and variance of X
- $\mu_Y \in \mathbb{R}$ and $\sigma_Y^2 \in (0, \infty)$ are the marginal mean and variance of Y
- $\rho \in [0, 1)$ is the correlation between X and Y

We write $(X,Y) \sim \text{biv Normal}(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$.

The marginal distributions of X and Y are

$$X \sim \text{Normal}(\mu_X, \sigma_X^2) \quad \text{and} \quad Y \sim \text{Normal}(\mu_Y, \sigma_Y^2).$$

Note that if $\rho = 0$, the joint pdf of (X,Y) may be written as the product of the marginal pdfs of X and Y:

$$f(x,y;\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho=0) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_X}\exp\left[-\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2}\right] \cdot \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_Y}\exp\left[-\frac{1}{2}\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right].$$

Thus, if (X,Y) are bivariate Normal with correlation equal to 0, then X and Y are independent.

* This is a special case in which $\text{corr}(X,Y) = 0$ implies independence of X and Y. Keep in mind that this does not hold in general.

The bivariate Normal distribution with $\mu_X = \mu_Y = 0$ and $\sigma_X^2 = \sigma_Y^2 = 1$ is called the standard bivariate Normal distribution.

If (Z_1, Z_2) has the standard bivariate Normal distribution then the joint pdf of (Z_1, Z_2) is given by

$$\phi(z_1, z_2; \rho) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(z_1^2 - 2\rho z_1 z_2 + z_2^2\right)\right].$$

If $(X,Y) \sim \text{biv Normal}(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$ then

$$(X,Y) \mapsto \left(\frac{X-\mu_X}{\sigma_X}, \frac{Y-\mu_Y}{\sigma_Y}\right) \sim \text{biv Normal}(0,0,1,1,\rho),$$

(standard bivariate Normal distribution)

and, likewise, if $(Z_1, Z_2) \sim \text{biv Normal}(0,0,1,1,\rho)$, then

$$(Z_1, Z_2) \mapsto (\mu_X + \sigma_X Z_1, \mu_Y + \sigma_Y Z_2) \sim \text{biv Normal}(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho).$$

ADDENDUM: To show that $\rho_{XY} \in [-1, 1]$, we will make use of the following:

Thm: (Cauchy-Schwarz inequality): For any r.v.s X and Y,

$$|\mathbb{E}XY| \leq \mathbb{E}|XY| \leq \sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2}.$$

Proof: • We first show that $|\mathbb{E}XY| \leq \mathbb{E}|XY|$.

Since $-|xy| \leq xy \leq |xy|$ for any $x, y \in \mathbb{R}$, we have if X and Y are continuous with joint pdf $f_{(X,Y)}$,

(If X, Y discrete, replace f with the joint pmf of X and Y and replace integrals with sums.)

$$\iint_{\mathbb{R}^2} -|xy|\, f(x,y)\, dx\, dy \leq \iint_{\mathbb{R}^2} xy\, f(x,y)\, dx\, dy \leq \iint_{\mathbb{R}^2} |xy|\, f(x,y)\, dx\, dy,$$

so that $-\mathbb{E}|XY| \leq \mathbb{E}XY \leq \mathbb{E}|XY|$, giving $|\mathbb{E}XY| \leq \mathbb{E}|XY|$.

(for any $x, a \in \mathbb{R}$, $a > 0$, $-a \leq x \leq a \iff |x| \leq a$)

• Now we show the second inequality $\mathbb{E}|XY| \leq \sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2}$.

We will use the fact that $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for all $a, b \in \mathbb{R}$, which we can get from

$$0 \leq (a-b)^2 = a^2 + b^2 - 2ab \iff 2ab \leq a^2 + b^2$$

(↑ a squared quantity is always non-negative)

$$\iff ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

Now, setting $a = \dfrac{|X|}{\sqrt{\mathbb{E}X^2}}$ and $b = \dfrac{|Y|}{\sqrt{\mathbb{E}Y^2}}$ we get

(for any $a, b \in \mathbb{R}$, $|a||b| = |ab|$)

$$\frac{|XY|}{\sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2}} \leq \frac{1}{2}\left(\frac{|X|}{\sqrt{\mathbb{E}X^2}}\right)^2 + \frac{1}{2}\left(\frac{|Y|}{\sqrt{\mathbb{E}Y^2}}\right)^2.$$

Taking expectations of both sides, we get

$$\frac{\mathbb{E}|XY|}{\sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2}} \leq 1$$

since $\mathbb{E}\left(\dfrac{|X|}{\sqrt{\mathbb{E}X^2}}\right)^2 = \dfrac{\mathbb{E}X^2}{\mathbb{E}X^2} = 1$ and $\mathbb{E}\left(\dfrac{|Y|}{\sqrt{\mathbb{E}Y^2}}\right)^2 = \dfrac{\mathbb{E}Y^2}{\mathbb{E}Y^2} = 1$.

This gives $\mathbb{E}|XY| \leq \sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2}$.

Corollary: The correlation ρ_{XY} between any two rvs X and Y is in the interval $[-1, 1]$.

Proof: Let $U = X - \mathbb{E}X$ and $V = Y - \mathbb{E}Y$. Then

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}} = \frac{\mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y)}{\sqrt{\mathbb{E}(X-\mathbb{E}X)^2}\sqrt{\mathbb{E}(Y-\mathbb{E}Y)^2}} = \frac{\mathbb{E}UV}{\sqrt{\mathbb{E}U^2}\sqrt{\mathbb{E}V^2}}.$$

So

$$|\rho_{XY}| = \frac{|\mathbb{E}UV|}{\sqrt{\mathbb{E}U^2}\sqrt{\mathbb{E}V^2}} \leq \frac{\mathbb{E}|UV|}{\sqrt{\mathbb{E}U^2}\sqrt{\mathbb{E}V^2}} \leq 1,$$

since by Cauchy-Schwarz $|\mathbb{E}UV| \leq \mathbb{E}|UV| \leq \sqrt{\mathbb{E}U^2}\sqrt{\mathbb{E}V^2}$.