STAT 511 fa 2019 Exam II

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- Do not open this test until told to do so.
- No calculators allowed; no notes allowed; no books allowed.
- Simplify all answers.
- SHOW YOUR WORK so that PARTIAL CREDIT may be given.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\text{Var}\,X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^{k=1}\frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

1. Let X be a rv with mgf given by

$$M_X(t) = \sum_{j=1}^{\infty}(t\theta)^{j-1}/j! = 1 + \frac{t\theta}{2} + \frac{t^2\theta^2}{3!} + \frac{t^3\theta^3}{4!} + \frac{t^4\theta^4}{5!} + \dots$$

for some $\theta > 0$.

   (a) Find $\mathbb{E}X$.

   (b) Find $\operatorname{Var} X$.

2. For each of the following, find the normalizing constant C (the constant required so that the pdf will integrate to 1). *Hint: Identify the type of distribution, i.e. Gamma, Beta, or Normal, and then identify the parameters; you should not have to do any integration.*

   (a) $f_Y(y) = Cy^3e^{-y/2}$ for $y > 0$.

   (b) $f_X(x) = Ce^{-x}$ for $x > 0$.

   (c) $f_Y(y) = Cy^3(1-y)$ for $0 < y < 1$.

   (d) $f_X(x) = Ce^{-x^2/2}$ for $-\infty < x < \infty$.

3. Let $(X, Y) \sim f(x, y) = \lambda^{-2}e^{-(x+y)/\lambda}$ for $x, y > 0$, where $\lambda > 0$.

   (a) Find an expression for the joint cdf $F(x, y) = P(X \leq x, Y \leq y)$ for all $x, y \in \mathbb{R}$.

   (b) Find the marginal pdf of X.

   (c) Give $P(X = \lambda)$.

4. Let

$$(X, Y) \sim f(x, y) = \begin{cases} 1 & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $\mathbb{E}(X - Y)^2$.

   (b) Find $P(X < Y)$.

   (c) Find $P(|X - Y| < 1/2)$. *Hint: Draw a picture of this region.*

5. Let $Y \sim f_Y(y) = 2y/\theta^2$ for $0 < y < \theta$, for some $\theta > 0$.

   (a) Find the median of Y in terms of θ.

   (b) Find the value of θ under which the 0.25 quantile is equal to 1.

   Hint: Set up the appropriate integrals and solve for the unknown quantity.

6. Let $X \sim \text{Poisson}(\lambda)$.

   (a) Give the second moment of X.

   (b) Suppose $\operatorname{Var} X = 10$. Find $P(X \geq 1)$.

7. Suppose $X \sim \text{Beta}(2, 2)$ and let $Y = 3 + 4X$.

   (a) Find $\mathbb{E}Y$.

   (b) Find $\operatorname{Var} Y$.