STAT 511 fa 2019 Exam II

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- Do not open this test until told to do so.
- No calculators allowed; no notes allowed; no books allowed.
- Simplify all answers.
- SHOW YOUR WORK so that PARTIAL CREDIT may be given.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\text{Var}\,X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^{k=1}\frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

1. Let X be a rv with mgf given by

$$M_X(t) = \sum_{j=1}^{\infty} (t\theta)^{j-1}/j! = 1 + \frac{t\theta}{2} + \frac{t^2\theta^2}{3!} + \frac{t^3\theta^3}{4!} + \frac{t^4\theta^4}{5!} + \dots$$

for some $\theta > 0$.

(a) Find $\mathbb{E}X$.

Solution: The first derivative of $M_X(t)$ with respect to t is

$$M_X^{(1)}(t) = \frac{\theta}{2} + \frac{2t\theta^2}{3!} + \frac{3t^2\theta^3}{4!} + \frac{4t^3\theta^4}{5!} + \dots,$$

which, when evaluated at $t = 0$ returns

$$\mathbb{E}X = \theta/2.$$

(b) Find $\operatorname{Var} X$.

Solution: The second derivative of $M_X(t)$ with respect to t is

$$M_X^{(1)}(t) = \frac{2\theta^2}{3!} + \frac{3(2)t\theta^3}{4!} + \frac{4(3)t^2\theta^4}{5!} + \dots,$$

which, when evaluated at $t = 0$ returns

$$\mathbb{E}X^2 = 2\theta^2/3! = \theta^2/3.$$

Then sing the useful expression $\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2$ gives

$$\operatorname{Var} X = \theta^2/3 - (\theta/2)^2 = (4\theta^2 - 3\theta^2)/12 = \theta^2/12.$$

2. For each of the following, find the normalizing constant C (the constant required so that the pdf will integrate to 1). *Hint: Identify the type of distribution, i.e. Gamma, Beta, or Normal, and then identify the parameters; you should not have to do any integration.*

(a) $f_Y(y) = Cy^3 e^{-y/2}$ for $y > 0$.

Solution: When normalized, this will become the pdf of the Gamma distribution with $\alpha = 4$ and $\beta = 2$. We have

$$C = \frac{1}{\Gamma(4)4^2} = 1/(3 * 2 * 16) = 1/(3 * 32) = 1/96.$$

(b) $f_X(x) = Ce^{-x}$ for $x > 0$.

Solution: When normalized, this will become the pdf of the Gamma distribution with $\alpha = 1$ and $\beta = 1$. We have

$$C = \frac{1}{\Gamma(1)1^1} = 1.$$

(c) $f_Y(y) = Cy^3(1-y)$ for $0 < y < 1$.

Solution: When normalized, this will become the pdf of the Beta distribution with $\alpha = 4$ and $\beta = 2$. We have

$$C = \frac{\Gamma(4+2)}{\Gamma(4)\Gamma(2)} = \frac{5!}{3!1!} = 20.$$

(d) $f_X(x) = Ce^{-x^2/2}$ for $-\infty < x < \infty$.

Solution: When normalized, this will become the pdf of the Normal distribution with $\mu = 0$ and $\sigma^2 = 1$. We have

$$C = \frac{1}{\sqrt{2\pi}}.$$

3. Let $(X, Y) \sim f(x, y) = \lambda^{-2}e^{-(x+y)/\lambda}$ for $x, y > 0$, where $\lambda > 0$.

(a) Find an expression for the joint cdf $F(x, y) = P(X \le x, Y \le y)$ for all $x, y \in \mathbb{R}$.

Solution: For $x > 0$ and $y > 0$ we have

$$\begin{aligned} P(X \le x, Y \le y) &= \int_0^x \int_0^y \frac{1}{\lambda^2} e^{-(t_1+t_2)/\lambda} dt_2 dt_1 \\ &= \int_0^x \frac{1}{\lambda} e^{-t_1/\lambda} \int_0^y \frac{1}{\lambda} e^{-t_2/\lambda} dt_2 dt_1 \\ &= (1 - e^{-y/\lambda}) \int_0^x \frac{1}{\lambda} e^{-t_1/\lambda} dt_1 \\ &= (1 - e^{-y/\lambda})(1 - e^{-x/\lambda}) \end{aligned}$$

So the joint cdf is given by

$$F(x, y) = \begin{cases} (1 - e^{-y/\lambda})(1 - e^{-x/\lambda}) & \text{for } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find the marginal pdf of X.

Solution: For $x > 0$, we have

$$f_X(x) = \int_0^\infty \frac{1}{\lambda^2} e^{-(x+y)/\lambda} dy = \frac{1}{\lambda} e^{-x/\lambda} \int_0^\infty \frac{1}{\lambda} e^{-y/\lambda} dy = \frac{1}{\lambda} e^{-x/\lambda}.$$

(c) Give $P(X = \lambda)$.

Solution: This is equal to zero because X is a continuous random variable.

4. Let

$$(X, Y) \sim f(x, y) = \begin{cases} 1 & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}(X - Y)^2$.

Solution: We have

$$\begin{aligned} \mathbb{E}(X - Y)^2 &= \int_0^1 \int_0^1 (x - y)^2 \cdot 1 \cdot dx dy \\ &= \int_0^1 \int_0^1 (x^2 - 2xy + y^2) dx dy \\ &= \int_0^1 (x^3/3 - x^2 y + xy^2) \Big|_0^1 dx dy \\ &= \int_0^1 (1/3 - y + y^2) dy \\ &= (y/3 - y^2/2 + y^3/3) \Big|_0^1 \\ &= 1/3 - 1/2 + 1/3 \\ &= 1/6. \end{aligned}$$

(b) Find $P(X < Y)$.

Solution: We can see that the answer is 1/2 by drawing a picture.

(c) Find $P(|X - Y| < 1/2)$. *Hint: Draw a picture of this region.*

Solution: We can see that the answer is 3/4 by drawing a picture.

5. Let $Y \sim f_Y(y) = 2y/\theta^2$ for $0 < y < \theta$, for some $\theta > 0$.

(a) Find the median of Y in terms of θ.

Solution: Solve the following for q:

$$1/2 = \int_0^q 2y/\theta^2 dy = y^2/\theta^2\Big|_0^q = q^2/\theta^2 \iff \theta^2/2 = q^2 \iff q = \theta/\sqrt{2},$$

so the median is $\theta/\sqrt{2}$.

(b) Find the value of θ under which the 0.25 quantile is equal to 1.

Solution: Solve the following for θ:

$$1/4 = \int_0^1 2y/\theta^2 dy = 1/\theta^2 \iff \theta = 2,$$

so under $\theta = 2$ the 0.25 quantile is equal to 1.

Hint: Set up the appropriate integrals and solve for the unknown quantity.

6. Let $X \sim \text{Poisson}(\lambda)$.

(a) Give the second moment of X.

Solution: By the useful expression and the fact that $\mathbb{E}X = \lambda$ and $\text{Var}\, X = \lambda$, we have $\lambda = \mathbb{E}X^2 - \lambda^2$, so

$$\mathbb{E}X^2 = \lambda(1 + \lambda).$$

(b) Suppose $\text{Var}\, X = 10$. Find $P(X \geq 1)$.

Solution: If $\text{Var}\, X = 10$ then $\lambda = 10$. Then

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{e^{-10}10^0}{0!} = 1 - e^{-10}.$$

7. Suppose $X \sim \text{Beta}(2, 2)$ and let $Y = 3 + 4X$.

(a) Find $\mathbb{E}Y$.

Solution: We have $\mathbb{E}X = 2/(2 + 2) = 1/2$, so

$$\mathbb{E}Y = 3 + 4\mathbb{E}X = 3 + 4/2 = 5.$$

(b) Find $\text{Var}\, Y$.

Solution: We have

$$\operatorname{Var} X = \frac{2 * 2}{(2+2)^2(2+2+1)} = \frac{4}{16 * 5} = \frac{4}{80} = \frac{1}{20}.$$

Then

$$\operatorname{Var} Y = 4^2 \operatorname{Var} X = 4^2/20 = 16/20 = 4/5.$$