STAT 511 fa 2019 Final Exam

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Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\operatorname{Var} X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0,1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$,	$x = 0,1,\dots,n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1,2,\dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0,1,\dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0,1,\dots,K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1,\dots,K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1,\dots,x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$p_{(X_1,\dots,X_K)}(x_1,\dots,x_K;p_1,\dots,p_K) = p_1^{x_1}\cdots p_K^{x_K}\cdot\mathbf{1}\left\{(x_1,\dots,x_K)\in\{0,1\}^K : \sum_{k=1}^K x_k = 1\right\}$$

$$p_{(Y_1,\dots,Y_K)}(y_1,\dots,y_K;n,p_1,\dots,p_K) = \left(\frac{n!}{y_1!\cdots y_K!}\right)p_1^{y_1}\cdots p_K^{y_K}\cdot\mathbf{1}\left\{(y_1,\dots,y_K)\in\{0,1,\dots,n\}^K : \sum_{k=1}^K y_k = n\right\}$$

$$f_{(X,Y)}(x,y;\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho) = \frac{1}{2\pi}\frac{1}{\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right)$$

Remember, remember! $\operatorname{Var}(aX+bY) = a^2\operatorname{Var}X + b^2\operatorname{Var}Y + 2ab\operatorname{Cov}(X,Y)$.

1. Customers of the pre-paid mobile provider *Talk-is-Cheap* will be randomly assigned to win a prize: 80% of the customers will receive 50 free text messages; 15% will receive 100 free text messages, and 5% will receive 200 free text messages. Suppose you are a *Talk-is-Cheap* customer and let X be the number of free text messages you receive.
   (a) Make a table showing the probability distribution of X.
   (b) Find $\mathbb{E}X$.
   (c) Give an expression for the cdf $F(x)$ of X for all $x \in \mathbb{R}$.
   (d) Draw a detailed picture of the cdf.

2. Hungry bear commutes to work either by biking or driving, and midmorning he sometimes eats a banana. On 80% of the days on which he bikes he eats a banana, and he eats a banana on 30% of the days on which he drives. He bikes on 90% of days.
   (a) On what proportion of days does hungry bear eat a banana midmorning?
   (b) If you see hungry bear eating a banana midmorning, what is the probability that he biked to work?
   (c) Are the events that hungry bear bikes to work and that he eats a banana independent?

3. Let X be a random variable with cdf given by

$$F(x) = \begin{cases} 1, & 9 \leq x \\ \sqrt{x}/3, & 0 \leq x < 9 \\ 0, & x < 0 \end{cases}$$

   (a) Give $P(X = 4)$.
   (b) Give $P(X > 2)$.
   (c) Find the pdf of X.
   (d) Find $\mathbb{E}X$.

4. Let $X \sim \text{Poisson}(5)$ and $Y \sim \text{Poisson}(6)$ and suppose X and Y are independent.
   (a) Find $\mathbb{E}(10X + 5)$.
   (b) Find $\text{Var}(10X + 5)$.
   (c) Find the distribution of $X + Y$ using the fact that $M_{X+Y}(t) = M_X(t)M_Y(t)$.
   (d) Find $\text{Var}(X + Y)$.
   (e) Write down the joint pmf of X and Y.

5. Let (U, V) be a pair of random variables with joint pdf given by

$$f(u, v) = \frac{2}{\pi} \exp\left(-\frac{u^2 + v^2}{2}\right) \qquad \text{for } u > 0,\ v > 0.$$

   (a) State whether U and V are independent and give your reasoning.

   (b) Give $\text{Cov}(U, V)$.

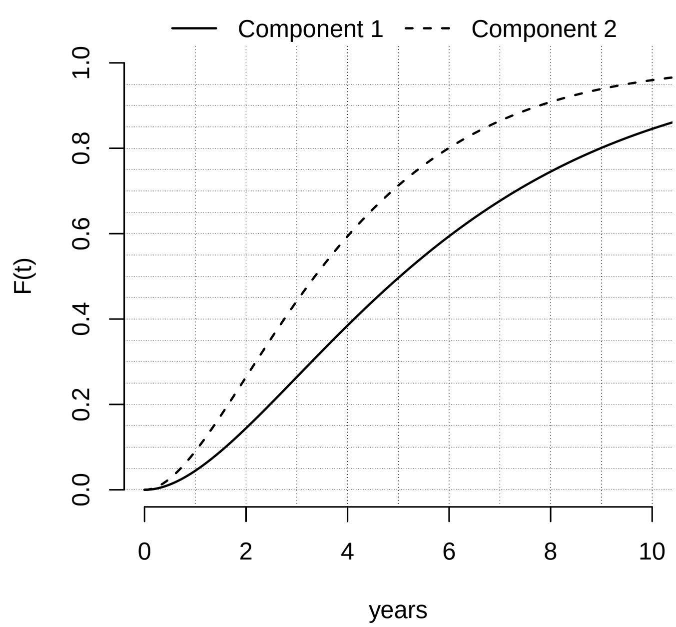
   (c) The marginal pdf of V is $f_V(v) = (\sqrt{2/\pi}) \exp(-v^2/2)$, for $v > 0$. Use this and the fact that U and V are independent to find the marginal pdf of U.

6. Suppose $X \sim \text{Normal}(2, 1)$ and $Y \sim \text{Normal}(0, 2)$ and $\text{Cov}(X, Y) = 1/2$.

   (a) Find $\mathbb{E}(3X + 4Y)$.

   (b) Find $\text{Var}(3X + 4Y)$.

7. Let T_1 and T_2 be independent random variables representing the times until failure of two components (component 1 and component 2, respectively) of a dishwasher. In order for the dishwasher to operate, both components must be functioning. The cdfs of T_1 and T_2 are plotted in the figure below.



   (a) What is the probability that component 1 fails in the first 2 years?

   (b) What is the probability that the dishwasher will still function after 6 years?

   (c) A 1-year warranty is offered with the dishwasher, under which the dishwasher will be replaced if *either* component fails during the 1-year period following the purchase. What is the probability that a customer may claim a replacement under the warranty?

8. Let X and Y be random variables such that

$$Y|X \sim \text{Uniform}(0, X) \quad \left(\quad \text{so } f(y|x) = \frac{1}{x}\mathbf{1}(0 < y < x) \quad \right)$$
$$X \sim \text{Exponential}(\lambda)$$

Note that the mean and variance of the Uniform(a, b)-dist. are $(a + b)/2$ and $(b - a)^2/12$, respectively.

(a) Find $\mathbb{E}(Y|X)$.

(b) Find $\text{Var}(Y|X)$.

(c) Find $\mathbb{E}Y$ using the fact that $\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|X])$.

(d) Find $\text{Var}\, Y$ using the fact that $\text{Var}\, Y = \mathbb{E}(\text{Var}[Y|X]) + \text{Var}(\mathbb{E}[Y|X])$.

(e) Write down the joint pdf of X and Y.

(f) Write down the integral you would need to solve in order to obtain the marginal pdf of Y (do not try to compute the integral).