STAT 511 fa 2019 Final Exam

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Do not open this test until told to do so; no calculators allowed; no notes allowed; no books allowed; show your work so that partial credit may be given.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\operatorname{Var} X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$$p_{(X_1,\dots,X_K)}(x_1,\dots,x_K;p_1,\dots,p_K) = p_1^{x_1}\cdots p_K^{x_K}\cdot \mathbf{1}\left\{(x_1,\dots,x_K)\in\{0,1\}^K : \textstyle\sum_{k=1}^K x_k = 1\right\}$$

$$p_{(Y_1,\dots,Y_K)}(y_1,\dots,y_K;n,p_1,\dots,p_K) = \left(\frac{n!}{y_1!\cdots y_K!}\right)p_1^{y_1}\cdots p_K^{y_K}\cdot \mathbf{1}\left\{(y_1,\dots,y_K)\in\{0,1,\dots,n\}^K : \textstyle\sum_{k=1}^K y_k = n\right\}$$

$$f_{(X,Y)}(x,y;\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho) = \frac{1}{2\pi}\frac{1}{\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right)$$

Remember, remember! $\operatorname{Var}(aX+bY) = a^2\operatorname{Var}X + b^2\operatorname{Var}Y + 2ab\operatorname{Cov}(X,Y)$.

1. Customers of the pre-paid mobile provider *Talk-is-Cheap* will be randomly assigned to win a prize: 80% of the customers will receive 50 free text messages; 15% will receive 100 free text messages, and 5% will receive 200 free text messages. Suppose you are a *Talk-is-Cheap* customer and let X be the number of free text messages you receive.

   (a) Make a table showing the probability distribution of X.

   Solution:

x	50	100	200
$P_X(X = x)$	0.80	0.15	0.05

   (b) Find $\mathbb{E}X$.

   Solution:

$$\mathbb{E}X = 50(0.80) + 100(0.15) + 200(0.05) = 40 + 15 + 10 = 65.$$

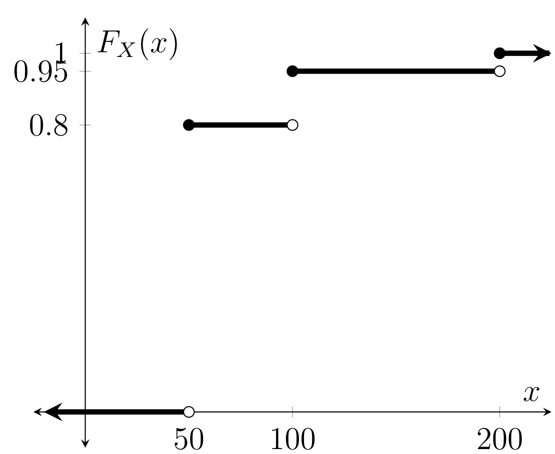
   (c) Give an expression for the cdf $F(x)$ of X for all $x \in \mathbb{R}$.

   Solution:

$$F(x) = \begin{cases} 1, & 200 \le x \\ 0.95, & 100 \le x < 200 \\ 0.80, & 50 \le x < 100 \\ 0, & x < 50. \end{cases}$$

   (d) Draw a detailed picture of the cdf.

   Solution:



2. Hungry bear commutes to work either by biking or driving, and midmorning he sometimes eats a banana.

On 80% of the days on which he bikes he eats a banana, and he eats a banana on 30% of the days on which he drives. He bikes on 90% of days.

(a) On what proportion of days does hungry bear eat a banana midmorning?

> **Solution:**
>
> $$\begin{aligned} P(\text{Banana}) &= P(\text{Banana} \cap \text{Bike}) + P(\text{Banana} \cap \text{Drive}) \\ &= P(\text{Banana}|\text{Bike})P(\text{Bike}) + P(\text{Banana}|\text{Drive})P(\text{Drive}) \\ &= 0.80(0.90) + (0.30)(0.10) \\ &= 0.75. \end{aligned}$$

(b) If you see hungry bear eating a banana midmorning, what is the probability that he biked to work?

> **Solution:**
>
> $$\begin{aligned} P(\text{Bike}|\text{Banana}) &= P(\text{Banana} \cap \text{Bike})/P(\text{Banana}) \\ &= P(\text{Banana}|\text{Bike})P(\text{Bike})/P(\text{Banana}) \\ &= 0.80(0.90)/0.75 \\ &= 0.96 \end{aligned}$$

(c) Are the events that hungry bear bikes to work and that he eats a banana independent?

> **Solution:** They are not independent since $P(\text{Banana}) = 0.75 \neq P(\text{Banana}|\text{Bike}) = 0.80$.

3. Let X be a random variable with cdf given by

$$F(x) = \begin{cases} 1, & 9 \leq x \\ \sqrt{x}/3, & 0 \leq x < 9 \\ 0, & x < 0 \end{cases}$$

(a) Give $P(X = 4)$.

> **Solution:** Since the cdf is continuous, X is a continuous random variable, so $P(X = 9) = 0$.

(b) Give $P(X > 2)$.

> **Solution:** $P(X > 2) = 1 - P(X \leq 2) = 1 - \sqrt{2}/3$.

(c) Find the pdf of X.

Solution: Taking the first derivative with respect to x of $F(x)$ for $0 < x < 9$ gives

$$f_X(x) = \frac{1}{6\sqrt{x}}\mathbf{1}(0 < x < 9).$$

(d) Find $\mathbb{E}X$.

Solution:

$$\mathbb{E}X = \int_0^9 x \cdot \frac{1}{6\sqrt{x}}dx = \frac{x^{3/2}}{9}\Big|_0^9 = 3.$$

4. Let $X \sim \text{Poisson}(5)$ and $Y \sim \text{Poisson}(6)$ and suppose X and Y are independent.

(a) Find $\mathbb{E}(10X + 5)$.

Solution: $\mathbb{E}(10X + 5) = 10\mathbb{E}X + 5 = 10(5) + 5 = 55$

(b) Find $\text{Var}(10X + 5)$.

Solution: $\text{Var}(10X + 5) = 100\,\text{Var}\,X = 100(5) = 500.$

(c) Find the distribution of $X + Y$ using the fact that $M_{X+Y}(t) = M_X(t)M_Y(t)$.

Solution: The mgf of $X + Y$ is given by

$$M_{X+Y}(t) = e^{5(e^t-1)}e^{6(e^t-1)} = e^{11(e^t-1)},$$

which is the mgf of the Poisson(11) distribution.

(d) Find $\text{Var}(X + Y)$.

Solution: Since $X + Y \sim \text{Poisson}(11)$, $\text{Var}(X + Y) = 11.$

(e) Write down the joint pmf of X and Y.

Solution: Since X and Y are independent, their joint pmf is given by

$$p(x, y) = \frac{e^{-5} \cdot 5^x}{x!} \cdot \frac{e^{-6} \cdot 6^y}{y!} \quad \text{for } x = 0, 1, 2, \ldots, \quad y = 0, 1, 2, \ldots$$

5. Let (U, V) be a pair of random variables with joint pdf given by

$$f(u,v) = \frac{2}{\pi} \exp\left(-\frac{u^2+v^2}{2}\right) \qquad \text{for } u > 0,\ v > 0.$$

(a) State whether U and V are independent and give your reasoning.

> **Solution:** They are independent because we can factor $f(u, v)$ as
>
> $$f(u,v) = \underbrace{\frac{2}{\pi} \exp\left(-\frac{u^2}{2}\right) \mathbf{1}(u > 0)}_{g(u)} \underbrace{\exp\left(-\frac{v^2}{2}\right) \mathbf{1}(v > 0)}_{h(v)},$$
>
> which is the product of a function of only u and a function of only v.

(b) Give $\text{Cov}(U, V)$.

> **Solution:** Since the random variables U and V are independent, $\text{Cov}(U, V) = 0$.

(c) The marginal pdf of V is $f_V(v) = \sqrt{2/\pi} \exp(-v^2/2)$, for $v > 0$. Use this and the fact that U and V are independent to find the marginal pdf of U.

> **Solution:** Since U and V are independent, the joint is the product of the marginals; that is, $f(u, v) = f_U(u) f_V(v)$. Therefore
>
> $$f_U(u) = f(u,v)/f_V(v) = \sqrt{2/\pi} \exp(-u^2/2).$$

6. Suppose $X \sim \text{Normal}(2, 1)$ and $Y \sim \text{Normal}(0, 2)$ and $\text{Cov}(X, Y) = 1/2$.

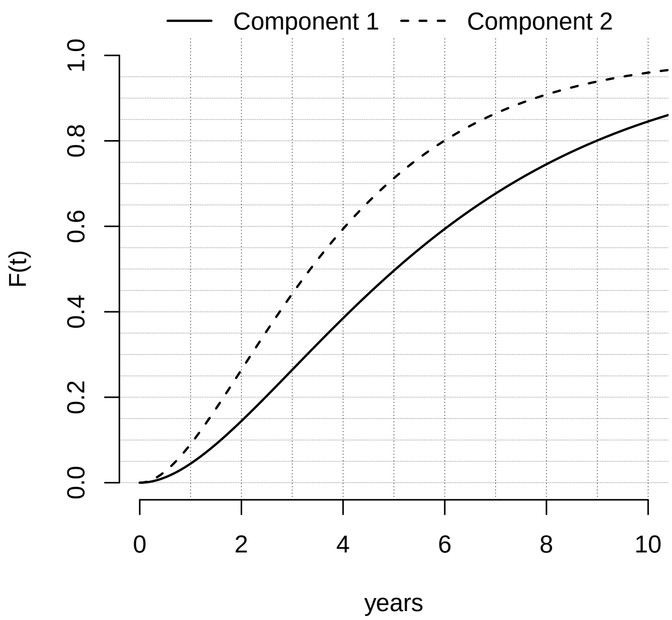
(a) Find $\mathbb{E}(3X + 4Y)$.

> **Solution:** $\mathbb{E}(3X + 4Y) = 3(2) + 4(0) = 6$.

(b) Find $\text{Var}(3X + 4Y)$.

> **Solution:** We have
>
> $$\begin{aligned} \text{Var}(3X+4Y) &= 9 \text{Var} X + 16 \text{Var} Y + 2(3)(4) \text{Cov}(X,Y) \\ &= 9(1) + 16(2) + 2(3)(4)(1/2) \\ &= 9 + 32 + 12 \\ &= 53. \end{aligned}$$

7. Let T_1 and T_2 be independent random variables representing the times until failure of two components (component 1 and component 2, respectively) of a dishwasher. In order for the dishwasher to operate, both components must be functioning. The cdfs of T_1 and T_2 are plotted in the figure below.



(a) What is the probability that component 1 fails in the first 2 years?

> **Solution:** This is $P(T_1 \leq 2) = 0.15$.

(b) What is the probability that the dishwasher will still function after 6 years?

> **Solution:** For the dishwasher to function after 6 years, the events $T_1 > 6$ and $T_2 > 6$ must both occur. We have
>
> $$P(T_1 > 6 \cap T_2 > 6) = P(T_1 > 6)P(T_2 > 6) = (1 - P(T_1 \leq 6))(1 - P(T_2 \leq 6)) = 0.4(0.2) = 0.08,$$
>
> where we have used the fact that T_1 and T_2 are independent.

(c) A 1-year warranty is offered with the dishwasher, under which the dishwasher will be replaced if *either* component fails during the 1-year period following the purchase. What is the probability that a customer may claim a replacement under the warranty?

Solution: We must find the probability of the event $P(T_1 \le 1 \cup T_2 \le 1)$. This is given by

$$\begin{aligned} P(T_1 \le 1 \cup T_2 \le 1) &= (P(T_1 \le 1) + P(T_2 \le 1) - P(T_1 \le 1 \cap T_2 \le 1) \\ &= 0.05 + 0.10 - 0.05(0.10 \\ &= 0.145. \end{aligned}$$

8. Let X and Y be random variables such that

$$\begin{aligned} Y|X &\sim \text{Uniform}(0, X) \quad \left(\quad \text{so } f(y|x) = \frac{1}{x}\mathbf{1}(0 < y < x) \quad \right) \\ X &\sim \text{Exponential}(\lambda) \end{aligned}$$

Note that the mean and variance of the Uniform(a, b)-dist. are $(a + b)/2$ and $(b - a)^2/12$, respectively.

(a) Find $\mathbb{E}(Y|X)$.

Solution: $\mathbb{E}(Y|X) = X/2$.

(b) Find $\text{Var}(Y|X)$.

Solution: $\text{Var}(Y|X) = X^2/12$.

(c) Find $\mathbb{E}Y$ using the fact that $\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|X])$.

Solution: $\mathbb{E}Y = \mathbb{E}(\mathbb{E}[Y|X]) = \mathbb{E}(X/2) = \lambda/2$.

(d) Find $\text{Var}\, Y$ using the fact that $\text{Var}\, Y = \mathbb{E}(\text{Var}[Y|X]) + \text{Var}(\mathbb{E}[Y|X])$.

Solution: We have

$$\begin{aligned} \text{Var}\, Y &= \mathbb{E}(X^2/12) + \text{Var}(X/2) \\ &= (1/12)(\text{Var}\, X + (\mathbb{E}X)^2) + (1/4)\,\text{Var}\, X \\ &= (1/12)(\lambda^2 + \lambda^2) + (1/4)\lambda^2 \\ &= (2/12)\lambda^2 + (3/12)\lambda^2 \\ &= (5/12)\lambda^2. \end{aligned}$$

(e) Write down the joint pdf of X and Y.

Solution:

$$f(x, y) = \frac{1}{x}\frac{1}{\lambda}e^{-x/\lambda} \quad \text{for } x > 0,\ 0 < y < x.$$

(f) Write down the integral you would need to solve in order to obtain the marginal pdf of Y (do not try to compute the integral).

Solution:

$$f_Y(y) = \int_y^{\infty} \frac{1}{x}\frac{1}{\lambda}e^{-x/\lambda}dx.$$