STAT 511 su 2020 hw 1

Basics of sets and probability

1. An experiment consists of flipping a coin and rolling a 6-sided die. Define the events

$$
\begin{aligned}
A &= \text{"coin toss is a head"} \\
B &= \text{"coin toss is a tail"} \\
C &= \text{"odd number rolled"} \\
D &= \text{"even number rolled"} \\
E &= \text{"number greater than 3 rolled"}
\end{aligned}
$$

(a) List all the points the sample space $\mathcal{S}$ for the experiment.

$$\mathcal{S} = \left\{ \begin{array}{cccccc} (H,\epsdice{1}), & (H,\epsdice{2}), & (H,\epsdice{3}), & (H,\epsdice{4}), & (H,\epsdice{5}), & (H,\epsdice{6}) \\ (T,\epsdice{1}), & (T,\epsdice{2}), & (T,\epsdice{3}), & (T,\epsdice{4}), & (T,\epsdice{5}), & (T,\epsdice{6}) \end{array} \right\}$$

(b) List the sample points in the set $A \cup C$.

$$A \cup C = \left\{ \begin{array}{cccccc} (H,\epsdice{1}), & (H,\epsdice{2}), & (H,\epsdice{3}), & (H,\epsdice{4}), & (H,\epsdice{5}), & (H,\epsdice{6}) \\ (T,\epsdice{1}), & & (T,\epsdice{3}), & & (T,\epsdice{5}) & \end{array} \right\}$$

(c) List the sample points in the set $A \cap C$.

$$A \cap C = \{ \ (H,\epsdice{1}), \ \ (H,\epsdice{3}), \ \ (H,\epsdice{5}) \ \}$$

(d) List the sample points in E^C.

$$E^C = \left\{ \begin{array}{ccc} (H,\epsdice{1}), & (H,\epsdice{2}), & (H,\epsdice{3}) \\ (T,\epsdice{1}), & (T,\epsdice{2}), & (T,\epsdice{3}) \end{array} \right\}$$

(e) List the sample points in $A \cap C \cap E$.

$$A \cap C \cap E = \{(H,\epsdice{5})\}$$

(f) List the sample points in $A \cap (C \cup E)$.

$$A \cap (C \cup E) = \{\ (H, \text{⚀}),\ \ (H, \text{⚂}),\ \ (H, \text{⚃}),\ \ (H, \text{⚄}),\ \ (H, \text{⚅})\ \}$$

(g) List the sample points in $(E \cap A) \cup (E \cap B)$.

Since $A^C = B$, we have

$$(E \cap A) \cup (E \cap B) = E = \left\{ \begin{array}{ccc} (H, \text{⚃}), & (H, \text{⚄}), & (H, \text{⚅}) \\ (T, \text{⚃}), & (T, \text{⚄}), & (T, \text{⚅}) \end{array} \right\}$$

(h) List the sample points in $(E^C \cap C) \cup (E^C \cap D)$.

Since $C^C = D$, we have

$$(E^C \cap C) \cup (E^C \cap D) = E^C = \left\{ \begin{array}{ccc} (H, \text{⚀}), & (H, \text{⚁}), & (H, \text{⚂}) \\ (T, \text{⚀}), & (T, \text{⚁}), & (T, \text{⚂}) \end{array} \right\}$$

(i) Give two pairs of events in which the events are disjoint.

The events A and B are disjoint and the events C and D are disjoint.

(j) Give two different partitions of $\mathcal{S}$ consisting of sets from among A, B, C, D, and E.

The pair events A and B, as these are complement events, is a partition of $\mathcal{S}$. The same is true of the pair of events C and D.

(k) How can we represent the set $A \cap B$?

The set $A \cap B$ contains no elements, so we can represent it with the empty set symbol $\emptyset$.

2. For two events A and B use our elementary set operations $\cup$, $\cap$, and the complement to give representations of the following:

(a) Both A and B occur.

$A \cap B$

(b) Neither A nor B occurs; give two representations of this.

$(A \cup B)^C$ or $A^C \cap A^C$.

(c) At least one of the events A and B occurs.

$A \cup B$

(d) At least one of the events A and B does not occur; give two representations of this.

$A^C \cup B^C$ or $(A \cap B)^C$

(e) One of the events A and B occurs but not the other.

$(A \cap B^C) \cup (B \cap A^C)$.

3. For some events $A_1, A_2, \ldots, A_n$, use $\bigcup_{i=1}^{n}$ and $\bigcap_{i=1}^{n}$ and the complement to give representations of the following:

(a) None of the events $A_1, \ldots, A_n$ occur.

$\bigcap_{i=1}^{n} A_i^C$ or $\left(\bigcup_{i=1}^{n} A_i\right)^C$

(b) All of the events $A_1, \ldots, A_n$ occur.

$\bigcap_{i=1}^{n} A_i$

(c) At least one of the events $A_1, \ldots, A_n$ occurs.

$\bigcup_{i=1}^{n} A_i$

(d) At least one of the events $A_1, \ldots, A_n$ does not occur.

$\bigcup_{i=1}^{n} A_i^C$ or $\left(\bigcap_{i=1}^{n} A_i\right)^C$

4. For some events $B_1, \ldots, B_n$, interpret the following in words:

(a) $\left(\bigcap_{i=1}^{n} B_i\right)^C$

Not all the events occur, or, at least one event does not occur.

(b) $\left(\bigcup_{i=1}^{n} B_i\right)^C$

None of the events occur.

(c) $\bigcup_{i=1}^{n} B_i$

At least one of the events occurs.

(d) $\bigcap_{i=1}^{n} B_i$

All of the events occur.

5. For two events A and B, draw Venn diagrams to illustrate the following identities:
   (a) $A = (A \cap B) \cup (A \cap B^C)$.
   (b) If $A \subset B$ then $A = A \cap B$.
   (c) If $A \subset B$ then $B = A \cup (B \setminus A)$.

6. Merit raises are to be given to two employees from among five candidate employees. Three of the employees are women and two are men. Define the following events:

$$A = \text{“at least one man is selected for a merit raise”}$$
$$B = \text{“one man and one woman are selected for a merit raise”}$$

   (a) List all the sample points in the sample space $\mathcal{S}$ of the experiment of selecting two employees to receive merit raises.
   Hint: denote the three women by W_1, W_2, and W_3 and the two men by M_1 and M_2.

$$\mathcal{S} = \left\{ \begin{array}{ccccc} W_1W_2 & W_1W_3 & W_1M_1 & W_1M_2 & W_2W_3 \\ W_2M_1 & W_2M_2 & W_3M_1 & W_3M_2 & M_1M_2 \end{array} \right\}$$

   (b) Use operations on A and B to express the event that two women receive merit raises.

A^C

   (c) Use operations on A and B to express the event that two men receive merit raises.

$A \cap B^C$

   (d) Interpret in words the event $A^C \cup B$.

The recipients of the merit raise are not both men.

   (e) If the recipients of merit raises are randomly selected such that each outcome in $\mathcal{S}$ is equally likely, with what probability will one woman and one man receive the raise?

$6/10 = 3/5$.

7. Assume that the distribution of blood types in the US is the following:

		antigens in RBCs			
		A	B	AB	O
Rh factor	−	0.063	0.015	0.006	0.066
	+	0.357	0.085	0.034	0.374

The following table indicates which recipient/donor combinations are possible:

		Donor							
		O−	O+	A−	A+	B−	B+	AB−	AB+
Recipient	O−	✓	.	.	.	.	.	.	.
	O+	✓	✓	.	.	.	.	.	.
	A−	✓	.	✓	.	.	.	.	.
	A+	✓	✓	✓	✓	.	.	.	.
	B−	✓	.	.	.	✓	.	.	.
	B+	✓	✓	.	.	✓	✓	.	.
	AB−	✓	.	✓	.	✓	.	✓	.
	AB+	✓	✓	✓	✓	✓	✓	✓	✓

(a) Find the probability that an individual randomly selected from the US population can donate blood to an $AB-$ recipient.

$0.066 + 0.063 + 0.015 + 0.006 = 0.15$

(b) Find the probability that an individual randomly selected from the US population can receive blood from an $AB-$ donor.

$0.006 + 0.034 = 0.04$

(c) Find the probability that an individual randomly selected from the US population is Rh+.

$0.357 + 0.085 + 0.034 + 0.374 = 0.85$

(d) Find the probability that an individual randomly selected from the US population is Rh+ or has the B antigen in his or her red blood cells.

$0.357 + 0.085 + 0.034 + 0.374 + 0.015 + 0.006 = 0.871$.

8. Consider flipping a coin four times and recording the sequence of heads and tails.

(a) Letting H denote "heads" and T denote "tails", list all elements in the sample space $\mathcal{S}$.

$$\mathcal{S} = \left\{ \begin{array}{ccccc} HHHH & HHHT & HHTT & TTTH & TTTT \\ & HHTH & TTHH & TTHT & \\ & HTHH & THTH & THTT & \\ & THHH & HTHT & HTTT & \\ & & THHT & & \\ & & HTTH & & \end{array} \right\}$$

(b) Assuming that each point in the sample space is equally likely, what is the probability of observing exactly three heads?

$4/16 = 1/4$.

(c) List the sample points in the event $A =$ "exactly two heads come up".

$$A = \{HHTT, TTHH, THTH, HTHT, THHT, HTTH\}$$

(d) Give the probability of A^C.

$P(A^C) = 1 - P(A) = 1 - 6/16 = 10/16 = 5/8$.

(e) Give the probability that at least two tails come up.

$11/16$.

9. Use the Kolmogorov axioms to show that for any two events A and B

$$P(A) = P(A \cap B) + P(A \cap B^C).$$

We have

$$\begin{aligned} P(A) &= P(A \cap \mathcal{S}) \quad (\text{since } A \cap \mathcal{S} = A) \\ &= P(A \cap (B \cup B^C)) \quad (\text{since } B \text{ and } B^C \text{ form a partition of } \mathcal{S}) \\ &= P((A \cap B) \cup (A \cap B^C)) \quad (\text{by the distributive property}) \\ &= P(A \cap B) + P(A \cap B^C) \quad (\text{by Kolmogorov axiom 3}). \end{aligned}$$

10. Suppose Kobe Bryant could make free throws with a success rate of 83.69%.

(a) Give a lower bound for the probability with which he would make 5 out of 5 free throws.

Letting $A_1, \dots, A_5$ be the events that he makes the free throws, respectively. Then we have

$$P(\bigcap_{i=1}^{5} A_i) \geq 1 - \sum_{i=1}^{5} P(A_i^C) = 1 - 5(1 - 0.8369) = 0.1845.$$

(b) Give a lower bound for the probability with which he would make 10 out of 10 free-throws.

Letting $A_1, \dots, A_10$ be the events that he makes the free throws, respectively. Then we have

$$P(\bigcap_{i=1}^{10} A_i) \geq 1 - \sum_{i=1}^{10} P(A_i^C) = 1 - 10(1 - 0.8369) = -0.631.$$

(c) Comment on the usefulness of these bounds.

The second bound gives a negative number, so it is useless, as we know that probabilities are bounded from below by zero. The first bound is useful.