STAT 511 su 2020 hw 2

Counting problems

For some problems it will be useful to use the function `choose()` *in R to compute large combinations. Type* `?choose` *to read the documentation. See also the function* `factorial()`.

1. A lift brings a skier from a lodge to the top of a mountain; from there, three slopes lead to a hut (hut 1) halfway down the mountain and two different slopes lead to a different hut (hut 2) halfway down the mountain. From each of these two huts there are four slopes leading back to the lodge.

   (a) In how many ways can the skier ski from the top of the mountain to the lodge?

   $$\# \text{ ways } = 3 \times 4 + 2 \times 4 = 20.$$

   (b) If the skier chooses one of the several ways at random such that each way is equally probable, with what probability will she choose one which passes by hut 1?

   $$P(\text{Pass by hut 1}) = \frac{3 \times 4}{20} = \frac{3}{5}.$$

2. A skier must wear gloves, a scarf, goggles, boots, a hat, and a jacket.

   (a) In how many different sequences can the skier don these articles?

   $$\# \text{ ways } = 6! = 720.$$

   (b) If the skier chooses a sequence at random in which to don these articles, with what probability does she don her jacket before her gloves?

   To each possible sequence there corresponds another possible sequence which is exactly the same except that the positions of the jacket and the gloves are switched. For example, these are two such sequences:

gloves	boots	hat	*jacket*	goggles	scarf
jacket	boots	hat	*gloves*	goggles	scarf

   We want to count only the sequence in which she dons her jacket before her gloves, so we do not count the first sequence. So we find that we need to remove $1/2$ of all the sequences, so that we have

   $$\# \text{ ways } = 6!/2 = 720/2 = 360.$$

The probability that she puts her jacket on before her gloves is

$$P(\text{Jacket on before gloves}) = \frac{360}{720} = \frac{1}{2}.$$

3. A group of 10 tired skiers is awarded 7 free cups of hot cocoa and 3 free beers.

   (a) In how many ways can the skiers distribute the free drinks such that each skier gets one drink?

   We can regard this as a partition; we split the 10 people into two groups, one of size 7 and the other of size 3. We have

   $$\#\text{ ways } = \frac{10!}{7!3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 10 \cdot 3 \cdot 4 = 120.$$

   (b) Suppose you are one skier among the 10 skiers; in how many of the possible ways to distribute the drinks do you, yourself, receive a beer?

   If I receive a beer, then the remaining 9 skiers must be partitioned into two groups, one of size 7, the number of hot cocoas, and one of size 2, the number of remaining beers. So we have

   $$\#\text{ ways } = \frac{9!}{7!2!} = \frac{9 \cdot 8}{2} = 36.$$

   (c) If one of the possible ways to distribute the drinks is chosen at random, with what probability will you receive a beer?

   We have

   $$P(\text{I recieve a beer}) = \frac{36}{120} = \frac{3}{10}.$$

   (d) If 2 skiers among the 10 do not drink beer, in how many ways can the drinks be distributed?

   The 2 skiers who do not drink beer are given hot cocoa, and the remaining 8 people are partitioned into two groups, one of size 5, which is the remaining number of hot cocoas, and one of size 3, which is the number of beers. So we have

   $$\#\text{ ways } = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 56.$$

4. Consider reordering the letters of "`bamboozle`".

   (a) Show that 90,720 unique sequences of letters are possible.

   There are 9 letters, which can be ordered in 9! ways, but since the letters "b" and "o" appear twice, some of these orderings will be redundant. To remove the redundant orderings coming from transposing the "b"s or the "o"s, we divide by the number of ways in which these can be ordered amongst themselves. So we have

   $$\#\text{ ways} = \frac{9!}{2!2!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 3 \cdot 2 = 90720.$$

   (b) If one of the possible reorderings is chosen at random, what is the probability that it contains the string "`bamboo`"?

   Fix "`bamboo`" as a single character in the string "`bamboozle`". Then there are 4 characters to order. So

   $$\#\text{ ways} = 4! = 24,$$

   giving

   $$P(\text{Contains } \texttt{bamboo}) = \frac{24}{90720} = 0.0002645503.$$

   (c) If one of the possible reorderings is chosen at random, what is the probability that it contains the string "`ooze`"?

   Fixing "`ooze`", the remaining letters are `bambl`, of which two are redundant. So

   $$\#\text{ ways} = \frac{6!}{2!} = 360,$$

   giving

   $$P(\text{Contains } \texttt{ooze}) = \frac{360}{90720} = 0.003968254.$$

5. In the game Heckmeck am Bratwurmeck, players begin each turn by rolling 8 dice. Each die is like an ordinary 6-sided die except that the "six" is replaced by the depiction of a smiling worm.

   (a) Find the probability of rolling 8 worms.

   There are 6^8 different outcomes in the experiment of rolling 8 dice that have 6 sides. In only one of these outcomes do all dice show the same side. So we have

   $$P(\text{Roll 8 worms}) = \frac{1}{6^8} = \frac{1}{1679616} = 5.953742 \times 10^{-07}.$$

(b) Find the number of ways to roll exactly 5 worms.
Hint: First you must choose which 5 from among the 8 dice to come up worms; then find the number of ways in which the remaining dice can come up not *worms.*

There are $\binom{8}{5}$ ways to choose which dice will come up worms, and then each of the remaining 3 dice must come up *not* worms, which can happen in 5^3 ways. So we have a total of

$$\text{\# ways to get exactly 5 worms} = \binom{8}{5} \times 5^3 = 7000.$$

(c) Show that the probability of rolling 5 worms can be expressed as

$$\binom{8}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^{8-5}$$

Using the answer to the previous question, we have

$$P(\text{Roll exactly 5 worms}) = \frac{\binom{8}{5}5^3}{6^8} = \binom{8}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^{8-5}.$$

(d) Find the probability of rolling at least one worm.
Hint: Try 1 minus the probability of the complement event.

The complement to rolling at least one worm is the event that no worms are rolled. The number of ways in which to get 8 non-worms is 5^8, so we have

$$P(\text{Roll no worms}) = \frac{5^8}{6^8} = 0.232568.$$

So the answer is

$$P(\text{At least one worm rolled}) = 1 - P(\text{Roll no worms}) = 1 - 0.232568 = 0.767432.$$

6. In the Midwestern card game of Euchre, four players play in teams of two, using a deck of 24 cards from which each player is dealt a hand of 5 cards. The 24-card deck is made by keeping only the 9s, the 10s, the face cards, and the aces.

(a) Find the probability that you are dealt the jack of hearts.

There are $\binom{24}{5}$ possible hands. If you have the jack of hearts, there are $\binom{23}{4}$ remaining ways to be dealt the other 4 cards in your hand. So we have

$$P(\text{Jack of hearts in hand}) = \frac{\binom{1}{1}\binom{23}{4}}{\binom{24}{5}} = \frac{23 \cdot 22 \cdot 21 \cdot 20/4!}{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20} = \frac{5}{24} = 0.2083333.$$

(b) Find the probability that you are dealt the jack of hearts and the jack of diamonds.

$$P(\text{Jack of hearts and jack of diamonds}) = \frac{\binom{1}{1}\binom{1}{1}\binom{22}{3}}{\binom{24}{5}} = \frac{5 \cdot 4}{24 \cdot 23} = 0.03623188.$$

(c) Find the probability that you are dealt two jacks of the same color.

You can be dealt red jacks or black jacks, so the answer is twice the previous answer:

$$P(\text{Two jacks of the same color}) = 2 \cdot \frac{\binom{1}{1}\binom{1}{1}\binom{22}{3}}{\binom{24}{5}} = 0.07246377.$$

(d) Find the probability that any of the four players is dealt two jacks of the same color.

There are four players from which to choose, so the answer is the previous answer times 4:

$$P(\text{Any player dealt two jacks of the same color}) = 4 \cdot 2 \cdot \frac{\binom{1}{1}\binom{1}{1}\binom{22}{3}}{\binom{24}{5}} = 0.2898551.$$

(e) Find the probability that you are dealt cards which are all hearts.

There are 6 hearts in the 24-card deck, so we have

$$P(\text{All hearts}) = \frac{\binom{6}{5}}{\binom{24}{5}} = \frac{6}{42504} = \frac{1}{7084} = 0.0001411632.$$

(f) Find the probability that you are dealt cards all of the same suit.

There are four suits to choose from, so the answer is 4 times the previous answer.

$$P(\text{All of same suit}) = 4\frac{\binom{6}{5}}{\binom{24}{5}} = \frac{24}{42504} = 0.0005646527.$$

(g) Find the probability that a team possesses in the hands of the two players all the cards of one suit.

The team possesses 10 cards in total. The number of ways in which the teams 10 cards could be dealt is $\binom{24}{10}$, and the number of ways in which these cards could be all of the same suit is $4 \cdot \binom{6}{6}\binom{18}{4}$. So we have

$$P(\text{A teams has all cards of a suit}) = \frac{4 \cdot \binom{6}{6}\binom{18}{4}}{\binom{24}{10}} = 0.006240899.$$

7. A curbside farmstand offers a basket of bounty (B.O.B.) consisting of 8 veggie items randomly selected from the farmer's inventory. Suppose you show up to the farmstand at the end of the day and the following items remain in the inventory:

4 squash, 5 bell peppers, 3 bulbs of garlic, 8 sweet potatoes.

(a) Find the probability that you get all of the bell peppers and all of the garlic bulbs in your B.O.B.

There are a total of $\binom{20}{8}$ ways to select the individual veggie items, so this will be the denominator. We have

$$P(\text{All bell peppers and all garlic bulbs}) = \frac{\binom{4}{0}\binom{5}{5}\binom{3}{3}\binom{8}{0}}{\binom{20}{8}} = \frac{1}{125970} = 7.938398 \times 10^{-06}.$$

(b) Find the probability that you get 2 of each veggie in your B.O.B.

$$P(\text{Two of each}) = \frac{\binom{4}{2}\binom{5}{2}\binom{3}{2}\binom{8}{2}}{\binom{20}{8}} = \frac{5040}{125970} = 0.04000953.$$

(c) Suppose you insist on getting all five bell peppers, allowing the farmer to choose your remaining 3 veggies from among the other items in the inventory. Find the probability that you get one of each of the remaining items in your B.O.B.

$$P(\text{One each of remaining}) = \frac{\binom{4}{1}\binom{3}{1}\binom{8}{1}}{\binom{15}{3}} = \frac{96}{455} = 0.210989.$$

8. *The canonical birthday problem*: Consider selecting n people at random, and assume that their birthdays are uniformly distributed (each day equally likely) over a 365-day year.

(a) Give an expression for the probability that no two people among the n people share a birthday.

Consider assigning birthdays to each of the n people. Each person can be assigned one of 365 possible days, so there are n^{365} ways to assign birthdays to the individuals.
Now consider assigning birthdays to each of the n people, but without repeating the same birthday after it has been assigned to someone, so that no two people will share a birthday. There are $365!/(365-n)!$ ways to do this.
So

$$P(\text{No two people share a birthday}) = \frac{365!/(365-n)!}{n^{365}}.$$

(b) Give the smallest size n of the group for which the probability that there is at least one shared birthday is greater than 0.95.
Hint: You will have to search for the answer trying different values of n*.*

We have

$$\begin{aligned} P(\text{At least one shared birthday}) &= 1 - P(\text{no two people share a birthday}) \\ &= 1 - \frac{365!/(365-n)!}{365^n} \\ &= 1 - \frac{365 \times \cdots \times (365-n+1)}{365^n}. \end{aligned}$$

We need to find the smallest n which satisfies

$$1 - \frac{365 \times \cdots \times (365-n+1)}{365^n} \geq 0.95,$$

which we can do using excel or R. In R, we have

```
for(n in 1:50){

  p <- 1 - prod((365:(365 - n + 1)))/365^n

  print(c(n,p))

}
```

for which some of the output is

```
[1] 43.0000000  0.9239229
[1] 44.0000000  0.9328854
[1] 45.0000000  0.9409759
```

```
[1] 46.0000000  0.9482528
[1] 47.0000000  0.9547744
[1] 48.000000  0.960598
[1] 49.0000000  0.9657796
```

so we see that $n = 47$ is sufficient.