STAT 511 su 2020 hw 3

Conditional probability, independence, rvs, the cdf

1. Let A and B be disjoint with $P(A) > 0$ and $P(B) > 0$. Are A and B independent? Explain.

2. Let $A \subset B$ with $P(A) > 0$ and $P(B) < 1$. Show that $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

3. Show that for any A and B with $P(B) > 0$, we have $P(A^c|B) = 1 - P(A|B)$.

4. At half-time of a basketball game, a spectator is selected to play a game: She will make two free throw attempts. If she makes neither free throw, she wins \$0. If she makes one out of the two free throws, she may draw one bill from a bag containing ten 1-dollar bills and five 5-dollar bills. If she makes both free throw attempts, she may draw two bills from the bag. Assume that her free throw attempts are independent and that she makes each free throw with probability 0.7.

   (a) Find $P(\text{she makes one free throw})$.

   (b) Find $P(\text{she wins \$1}|\text{she makes one free throw})$.

   (c) Find $P(\text{she wins \$2}|\text{she makes two free throws})$.

   (d) Find $P(\text{she wins \$5 or more})$.

   (e) Find $P(\text{she made two free throws}|\text{she won \$5 or more})$.

5. A Bernoulli trial is an experiment with the sample space

$$\mathcal{S} = \{\text{success}, \text{failure}\}.$$

   Let X be the random variable on $\mathcal{S}$ defined by

$$X(s) = \begin{cases} 1 & \text{if } s = \text{success} \\ 0 & \text{if } s = \text{failure}. \end{cases}$$

   Suppose the probability $P(\{\text{success}\})$ is equal to some value $p \in (0, 1)$.

   (a) Tabulate for the rv* X the probability distribution $P_X(X = x)$ in terms of p for $x \in \{0, 1\}$.

   (b) Give the expression for the cdf† F_X of X (Remember that $F_X(x)$ must be defined for all $x \in \mathbb{R}$).

   (c) Make an accurate drawing of F_X when $p = 1/4$.

   (d) State whether X is a discrete or a continuous rv and why.

6. Suppose 4 Bernoulli trials, each with success probability p, are conducted such that the outcomes of the 4 experiments are mutually independent. Let the random variable X be the total number of successes over the 4 Bernoulli trials.

   (a) Write down the sample space for the experiment consisting of 4 Bernoulli trials (the sample space is all possible sequences of length 4 of successes and failures—you may use the symbols `S` and `F`).

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*rv = random variable

†cdf = cumulative distribution function

(b) Give the support (range) $\mathcal{X}$ of X.

(c) Tabulate for X the probability distribution $P_X(X = x)$ for $x \in \mathcal{X}$ in terms of the success probability p.

(d) Give the cdf F_X of X (Remember that $F_X(x)$ must be defined for all $x \in \mathbb{R}$) in terms of the success probability p.

(e) Make an accurate drawing of the cdf F_X of X in the case $p = 1/2$.

(f) State whether X is a discrete or a continuous rv and why.

7. Consider the function given by

$$F(x) = \begin{cases} 1 & \text{if} \quad 1 \leq x \\ x & \text{if} \quad 0 \leq x < 1 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

(a) Draw an accurate picture of the function F.

(b) Verify that F is a cdf (three properties to verify).

(c) Let X be a random variable with cdf $F_X = F$.

i. State whether X is a discrete or a continuous rv and why.

ii. Give the support (range) $\mathcal{X}$ of X.

iii. Give $P_X(X \leq 1/2)$

iv. $P_X(1/4 < X \leq 1/2)$

8. Consider rolling two K-sided dice with faces numbered $1, \dots, K$ and recording the outcome as $(\text{face } 1, \text{face } 2)$. Let X be the rv defined as the maximum of the two faces.

(a) Give the support $\mathcal{X}$ of the rv X.

(b) The sample space is

$$\mathcal{S} = \left\{ \begin{array}{ccccc} (1,1) & (1,2) & (1,3) & \dots & (1,K) \\ (2,1) & (2,2) & (2,3) & \dots & (2,K) \\ (3,1) & (3,2) & (3,3) & \dots & (3,K) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (K,1) & (K,2) & (K,3) & \dots & (K,K) \end{array} \right\}$$

Give the probability that $X = 2$.

(c) Suppose $K = 5$. List all the points in the sample space.

(d) Suppose $K = 5$. Finish tabulating the values of the probability distribution $P_X(X = x)$ for all $x \in \mathcal{X}$ (make sure your probabilities sum to 1):

x	1	2	$\cdots$
$P_X(X = x)$	$1/25$	$3/25$	$\cdots$

(e) Suppose $K = 5$. Give the cdf F_X of X.

(f) For a general number of sides K, the probabilities $P_X(X = x)$ are given by one of the following functions in x. Which is it?

A. $2x/(K(K+1))$, $x = 1, \ldots, K$

B. $(2x-1)/K^2$, $x = 1, \ldots, K$

C. $\binom{K}{x} / \sum_{x=1}^{K} \binom{K}{x}$, $x = 1, \ldots, K$

D. $(2(K-x+1)-1)/K^2$, $x = 1, \ldots, K$

(g) Suppose $K = 20$. Compute the probability that $X \geq 13$ (you may use software) and explain your calculations.

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 2.77
- 2.86, 2.90
- 2.134, 2.137