# STAT 511 su 2020 hw 3

*Conditional probability, independence, rvs, the cdf*

1. Let $A$ and $B$ be disjoint with $P(A) > 0$ and $P(B) > 0$. Are $A$ and $B$ independent? Explain.

Assume (in order that a contradiction will arise) that $A$ and $B$ *are* independent. Then $P(A \cap B) = P(A)P(B)$, which implies $P(A \cap B) > 0$, since $P(A) > 0$ and $P(B > 0)$. However, since $A$ and $B$ are disjoint, $P(A \cap B) = 0$, which is a contradiction. Therefore, $A$ and $B$ are *not* independent.

2. Let $A \subset B$ with $P(A) > 0$ and $P(B) < 1$. Show that $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

Note that $A \subset B \implies A \cap B = A$, so we have

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(B)} = 1$$

and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

3. Show that for any $A$ and $B$ with $P(B) > 0$, we have $P(A^c|B) = 1 - P(A|B)$.

We have

$$P(A^c|B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = 1 - \frac{P(A \cap B)}{P(B)} = 1 - P(A|B).$$

4. At half-time of a basketball game, a spectator is selected to play a game: She will make two free throw attempts. If she makes neither free throw, she wins \$0. If she makes one out of the two free throws, she may draw one bill from a bag containing ten 1-dollar bills and five 5-dollar bills. If she makes both free throw attempts, she may draw two bills from the bag. Assume that her free throw attempts are independent and that she makes each free throw with probability 0.7.

   (a) Find $P(\text{she makes one free throw})$.

The sample space for the part of the experiment involving the free throws is

$$\mathcal{S} = \{11, 10, 01, 00\},$$

letting 1 denote making the free throw and 0 denote missing the free throw. These sample points occur with the probabilities $(0.7)^2$, $(0.7)(0.3)$, $(0.3)(0.7)$, and $(0.3)(0.3)$, respectively. The middle two sample points correspond to making one free throw, so we have

$$P(\text{she makes one free throw}) = (0.7)(0.3) + (0.3)(0.7) = 2(0.7)(0.3) = 0.42.$$

(b) Find $P(\text{she wins \$1}|\text{she makes one free throw})$.

If she makes one free throw, she draws one bill from the bag. She must draw one bill from among the ten \$1-bills and none from among the five \$5-dollar bills. There are a total of 15 bills in the bag, so there are 15 possible bills she could draw. So we have

$$P(\text{she wins \$1}|\text{she makes one free throw}) = \frac{10}{15} = \frac{2}{3}.$$

(c) Find $P(\text{she wins \$2}|\text{she makes two free throws})$.

If she makes two free throws she will draw twice from the bag. To win \$2 she must draw two from among the ten \$1-dollar bills and none from among the five \$5-dollar bills. There are a total of $\binom{15}{2}$ ways for her to draw two bills from the bag. So we have

$$P(\text{she wins \$2}|\text{she makes two free throws}) = \frac{\binom{10}{2}\binom{5}{0}}{\binom{15}{2}} = \frac{10 \cdot 9}{15 \cdot 14} = \frac{9}{21} = 0.4285714.$$

(d) Find $P(\text{she wins \$5 or more})$.

The possible ways (drawing a picture helps you to see this) in which she could make \$5 or more are

$$\begin{aligned} \{\text{wins \$6}\} &= \{\text{makes 2 }\} \cap \{\text{draws a \$5-dollar and a \$1-dollar bill}\} \\ \{\text{wins \$10}\} &= \{\text{makes 2}\} \cap \{\text{draws two \$5-dollar bills}\} \\ \{\text{wins \$5}\} &= \{\text{makes 1}\} \cap \{\text{draws a \$5-dollar bill}\}, \end{aligned}$$

so that

$$
\begin{aligned}
P(\text{wins } \$6) &= P(\text{draws a } \$5\text{-dollar and a } \$1\text{-dollar bill}|\text{makes 2 }) \cdot P(\text{makes 2 }) \\
&= \frac{\binom{10}{1}\binom{5}{1}}{\binom{15}{2}} \cdot (0.7)^2 \\
&= \frac{50}{105}\frac{49}{100} \\
&= 49/210 \\
P(\text{wins } \$10) &= P(\text{draws two } \$5\text{-dollar bills}|\text{makes 2})P(\text{makes 2}) \\
&= \frac{\binom{10}{0}\binom{5}{2}}{\binom{15}{2}} \cdot (0.7)^2 \\
&= \frac{10}{105}\frac{49}{100} \\
&= 49/1050 \\
P(\text{wins } \$5) &= P(\text{draws a } \$5\text{-dollar bill}|\text{makes 1})P(\text{makes 1}) \\
&= \frac{\binom{10}{0}\binom{5}{1}}{\binom{15}{1}} \cdot 2(0.7)(0.3) \\
&= \frac{5}{15}\frac{42}{100} \\
&= 42/300 \\
&= 14/100.
\end{aligned}
$$

So the answer is

$$P(\text{she wins } \$5 \text{ or more}) = 49/210 + 49/1050 + 14/100 = 0.42.$$

(e) Find $P(\text{she made two free throws}|\text{she won } \$5 \text{ or more})$.

We have

$$
\begin{aligned}
P(\text{made two free throws}|\text{won } \$5 \text{ or more}) &= \frac{P(\text{made two free throws} \cap \text{won } \$5 \text{ or more})}{P(\text{won } \$5 \text{ or more})} \\
&= \frac{49/210 + 49/1050}{0.42} \\
&= 2/3.
\end{aligned}
$$

5. A Bernoulli trial is an experiment with the sample space

$$\mathcal{S} = \{\text{success}, \text{failure}\}.$$

Let $X$ be the random variable on $\mathcal{S}$ defined by

$$X(s) = \begin{cases} 1 & \text{if } s = \text{success} \\ 0 & \text{if } s = \text{failure}. \end{cases}$$

Suppose the probability $P(\{\text{success}\})$ is equal to some value $p \in (0,1)$.

(a) Tabulate for the rv* $X$ the probability distribution $P_X(X = x)$ in terms of $p$ for $x \in \{0,1\}$.

| $x$ | 0 | 1 |
|---|---|---|
| $P_X(X = x)$ | $1 - p$ | $p$ |

(b) Give the expression for the cdf† $F_X$ of $X$ (Remember that $F_X(x)$ must be defined for all $x \in \mathbb{R}$).

We have

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - p, & 0 \leq x < 1 \\ 1, & 1 \leq x \end{cases}$$

(c) Make an accurate drawing of $F_X$ when $p = 1/4$.

(d) State whether $X$ is a discrete or a continuous rv and why.

The random variable $X$ is discrete since its cdf is a step function.

6. Suppose $4$ Bernoulli trials, each with success probability $p$, are conducted such that the outcomes of the $4$ experiments are mutually independent. Let the random variable $X$ be the total number of successes over the $4$ Bernoulli trials.

(a) Write down the sample space for the experiment consisting of $4$ Bernoulli trials (the sample space is all possible sequences of length $4$ of successes and failures—you may use the symbols `S` and `F`).

We have

$$\mathcal{S} = \left\{ \begin{array}{ccccc} \mathtt{SSSS} & \mathtt{SSSF} & \mathtt{SSFF} & \mathtt{FFFS} & \mathtt{FFFF} \\ & \mathtt{SSFS} & \mathtt{FFSS} & \mathtt{FFSF} & \\ & \mathtt{SFSS} & \mathtt{FSFS} & \mathtt{FSFF} & \\ & \mathtt{FSSS} & \mathtt{SFSF} & \mathtt{SFFF} & \\ & & \mathtt{FSSF} & & \\ & & \mathtt{SFFS} & & \end{array} \right\}$$

*rv = random variable

†cdf = cumulative distribution function

(b) Give the support (range) $\mathcal{X}$ of $X$.

The support is $\mathcal{X} = \{0, 1, 2, 3, 4\}$.

(c) Tabulate for $X$ the probability distribution $P_X(X = x)$ for $x \in \mathcal{X}$ in terms of the success probability $p$.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P_X(X = x)$ | $(1-p)^4$ | $4p(1-p)^3$ | $6p^2(1-p)^2$ | $4p^3(1-p)$ | $p^4$ |

(d) Give the cdf $F_X$ of $X$ (Remember that $F_X(x)$ must be defined for all $x \in \mathbb{R}$) in terms of the success probability $p$.

$$F_X(x) = \begin{cases} 0 & x < 0 \\ (1-p)^4 & 0 \leq x < 1 \\ (1-p)^4 + 4p(1-p)^3 & 1 \leq x < 2 \\ (1-p)^4 + 4p(1-p)^3 + 6p^2(1-p)^2 & 2 \leq x < 3 \\ (1-p)^4 + 4p(1-p)^3 + 6p^2(1-p)^2 + 4p^3(1-p) & 3 \leq x < 4 \\ 1 & 4 \leq x \end{cases}$$

(e) Make an accurate drawing of the cdf $F_X$ of $X$ in the case $p = 1/2$.

(f) State whether $X$ is a discrete or a continuous rv and why.

It is a discrete rv because its cdf is a step function.

7. Consider the function given by

$$F(x) = \begin{cases} 1 & \text{if} \quad 1 \leq x \\ x & \text{if} \quad 0 \leq x < 1 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

(a) Draw an accurate picture of the function $F$.

(b) Verify that $F$ is a cdf (three properties to verify).

For $x$ approaching $-\infty$ the cdf approaches $0$, and for $x$ approaching $\infty$ the cdf approaches $1$. Also it is non-decreasing. This cdf is continuous, so it satisfies right-continuity.

(c) Let $X$ be a random variable with cdf $F_X = F$.

i. State whether $X$ is a discrete or a continuous rv and why.

It is continuous, because the cdf is a continuous function.

ii. Give the support (range) $\mathcal{X}$ of $X$.

The support could be any of the intervals $(0,1)$, $[0,1]$, $[0,1)$, or $(0,1]$. For any interval $(a,b)$, $a<b$, such that $(a,b)\cap(0,1)$ is non-empty, the cdf assigns positive probability. For intervals outside of $(0,1)$, the cdf assigns probability equal to zero.

iii. Give $P_X(X \leq 1/2)$

This is $F_X(1/2) = 1/2$.

iv. $P_X(1/4 < X \leq 1/2)$

This is $F_X(1/2) - F_X(1/4) = 1/2 - 1/4 = 1/4$.

8. Consider rolling two $K$-sided dice with faces numbered $1,\ldots,K$ and recording the outcome as $(\text{face } 1, \text{face } 2)$. Let $X$ be the rv defined as the maximum of the two faces.

(a) Give the support $\mathcal{X}$ of the rv $X$.

The support of $X$ is

$$\mathcal{X} = \{1,\ldots,K\}.$$

(b) The sample space is

$$\mathcal{S} = \left\{ \begin{array}{ccccc} (1,1) & (1,2) & (1,3) & \ldots & (1,K) \\ (2,1) & (2,2) & (2,3) & \ldots & (2,K) \\ (3,1) & (3,2) & (3,3) & \ldots & (3,K) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (K,1) & (K,2) & (K,3) & \ldots & (K,K) \end{array} \right\}$$

Give the probability that $X = 2$.

We have $X=2$ for the outcomes $(2,1)$, $(2,2)$, and $(1,2)$, so we have

$$P_X(X=2) = \frac{3}{K^2}.$$

(c) Suppose $K = 5$. List all the points in the sample space.

$$\mathcal{S} = \left\{ \begin{array}{ccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) \end{array} \right\}$$

(d) Suppose $K = 5$. Finish tabulating the values of the probability distribution $P_X(X = x)$ for all $x \in \mathcal{X}$ (make sure your probabilities sum to 1):

| $x$ | 1 | 2 | $\cdots$ |
|---|---|---|---|
| $P_X(X = x)$ | 1/25 | 3/25 | $\cdots$ |

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P_X(X = x)$ | 1/25 | 3/25 | 5/25 | 7/25 | 9/25 |

(e) Suppose $K = 5$. Give the cdf $F_X$ of $X$.

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 1/25 & 1 \leq x < 2 \\ 4/25 & 2 \leq x < 3 \\ 9/25 & 3 \leq x < 4 \\ 16/25 & 4 \leq x < 5 \\ 1 & 5 \leq x \end{cases}$$

(f) For a general number of sides $K$, the probabilities $P_X(X = x)$ are given by one of the following functions in $x$. Which is it?

A. $2x/(K(K+1))$, $x = 1, \ldots, K$

**B.** $(2x - 1)/K^2$**,** $x = 1, \ldots, K$

C. $\binom{K}{x} / \sum_{x=1}^{K} \binom{K}{x}$, $x = 1, \ldots, K$

D. $(2(K - x + 1) - 1)/K^2$, $x = 1, \ldots, K$

(g) Suppose $K = 20$. Compute the probability that $X \geq 13$ (you may use software) and explain your calculations.

We have

$$
\begin{aligned}
P(X \geq 13) &= 1 - P(X \leq 12) \\
&= 1 - \sum_{x=1}^{12} \frac{2x-1}{20^2} \\
&= 1 - \frac{1}{20^2} \sum_{x=1}^{12} (2x-1) \\
&= 1 - \frac{1}{20^2} \left[ 2 \frac{12(12+1)}{2} - 12) \right] \qquad \left( \text{using } \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right) \\
&= 1 - \left( \frac{12}{20} \right)^2 \\
&= 1 - \left( \frac{3}{5} \right)^2 \\
&= \frac{16}{25}.
\end{aligned}
$$

This could also be done with R using

`sum( 2*(13:20) - 1)/20**2` $= 0.64.$

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 2.77
- 2.86, 2.90
- 2.134, 2.137