STAT 511 su 2020 hw 4

pmfs, pdfs, expected values, variance

At some point in this assignment you will need to use the identities

$$\sum_{i=1}^{n} i = n(n+1)/2, \quad \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6, \quad \sum_{i=1}^{n} i^3 = n^2(n+1)^2/4.$$

1. Consider being dealt a hand of 5 cards from a 24-card Euchre deck, which has 9s, 10s, face cards, and aces in each of the four suits. Let the random variable X be the number of hearts in your hand.
   (a) Tabulate the probability distribution of X in a table like

x	$\dots$
$P_X(X = x)$	$\dots$

   (b) Find $P(X \leq 2)$.
   (c) Find $\mathbb{E}X$.
   (d) Find $\operatorname{Var} X$.

2. The random variable Y represents the amount of flooding over a baseline equal to 1 in a certain city. It has the cdf

$$F_Y(y) = \begin{cases} 1 - \frac{1}{y^2}, & 1 \leq y < \infty \\ 0, & -\infty < y < 1 \end{cases}$$

   (a) Draw an accurate picture of F_Y.
   (b) Verify that F_Y is a cdf (3 properties to verify).
   (c) Find $P_Y(1 < Y < 2)$.
   (d) Find the pdf f_Y of Y. *Hint: consider separately the intervals* $-\infty < y < 1$ *and* $1 \leq y < \infty$.
   (e) Draw an accurate picture of f_Y.
   (f) Find the value M such that $P(Y \leq M) = 1/2$. The value M is the median of Y.
   (g) Find $\mathbb{E}Y$.
   (h) Find $\operatorname{Var} Y$.
   (i) Are we able to apply Chebychev's inequality to this random variable in order to find an interval within which it will lie with probability exceeding some lower bound? Explain why or why not.

3. Let X be a random variable with pdf given by

$$f_X(x) = Cx^2(1 - x)\mathbf{1}(0 < x < 1),$$

where $C > 0$ and $\mathbf{1}(\cdot)$ is the indicator function.

(a) Find the value of the constant C such that f_X is a valid pdf.

(b) Find $P(1/2 \leq X < 1)$.

(c) Find $P(X \leq 1/2)$.

(d) Find $P(X = 1/2)$.

(e) Find $P(1 \leq X \leq 2)$.

(f) Find $\mathbb{E}X$.

4. For any rv X, verify the identity $\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2$.

5. Let X be a rv such that $\mathbb{E}X = \mu$ and $\operatorname{Var} X = \sigma^2$.

   (a) Show that $\operatorname{Var}(aX + b) = a^2\sigma^2$.

   (b) Show that $\mathbb{E}(aX + b) = a\mu + b$.

6. Consider rolling a single K-sided die and let X be the rv defined as the number on the up-face.

   (a) Give the pmf of the rv X.

   (b) Find $\mathbb{E}X$.

   (c) Find $\operatorname{Var} X$.

   (d) Give the expected value of the roll of a 6-sided die and interpret this quantity in your own words.

7. Consider rolling two K-sided dice with faces numbered $1, \ldots, K$ and let X be the rv defined as the maximum of the two up-faces. The pmf of X is given by

$$p_X(x) = (2x - 1)/K^2, \quad x = 1, \ldots, K.$$

   (a) Show that $p_X(x)$ sums to 1 over the support of X.

   (b) Find $\mathbb{E}X$.

   (c) Find $\mathbb{E}X^2$.

   (d) Give an expression for $\operatorname{Var} X$ (does not have to be simplified) and compute $\operatorname{Var} X$ for $K = 10$.

   (e) Let $K = 10$ and use Chebyschev's theorem to find an interval within which X will fall with probability at least $8/9$.

   (f) Comment on the usefulness of the interval computed in the previous part.

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 3.12, 3.24, 3.30
- 4.6 (*Hint: Shown in Lec 04*), 4.7, 4.8, 4.11, 4.13, 4.16