STAT 511 su 2020 hw 5

common pdfs, pmfs, expected value, variance

1. Let X have cdf given by

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b < x, \end{cases}$$

for some a and b with $a < b$.

(a) Write down the pdf of X.

(b) Draw a picture of the pdf of X.

(c) Give the support of X.

(d) Show that $\mathbb{E}X = (a+b)/2$.

(e) Show* that $\operatorname{Var} X = (a-b)^2/12$.

(f) Find $P(c < X < d)$ for any $c, d \in (a, b)$ with $c < d$.

(g) Let $Y = (X-a)/(b-a)$ and find

i. $\mathbb{E}Y$.

ii. $\operatorname{Var} Y$.

(h) Find $P(1/4 < Y < 3/4)$.

2. Let $X \sim \text{Beta}(\alpha, \beta)$.

(a) Show that

$$\mathbb{E}X^2 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}.$$

(b) Use the fact that $\mathbb{E}X = \alpha/(\alpha+\beta)$ and your answer to the previous part to show that

$$\operatorname{Var} X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

(c) Suppose X is the proportion of free-throws made over the lifetime of a randomly sampled kid, and assume that $X \sim \text{Beta}(2, 8)$.

i. Draw a picture of the pdf of X (you may use software if you wish).

ii. Compute the probability that the randomly selected kid has made fewer than 0.10 of all free throws. *Hint: Use the R function* `pbeta()`. *Access documention with* `?pbeta`.

iii. If you were to sample 100 kids and compute the average of their lifetime free-throw success proportions, to what number would you expect this average to be close?

(d) Let $\alpha = 1$ and $\beta = 1$. For some constants a and b with $a < b$, let $Y = (b-a)X + a$. Find

*A useful fact: $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$ for any x, y.

   i. the support of the new random variable Y.
   ii. $\mathbb{E}Y$.
   iii. $\operatorname{Var} Y$.

(e) Give the pdf of X under $\alpha = 1$ and $\beta = 1$.

(f) Give the cdf of X under $\alpha = 1$ and $\beta = 1$.

(g) What is the relationship between the beta distribution and the uniform distribution?

3. Let $X \sim \text{Exponential}(\lambda)$ for some $\lambda > 0$.

   (a) Show that for any integer $k > 0$

   $$\mathbb{E}X^k = \lambda^k k!$$

   Hint: Set up the integral, do a change of variable $u = x/\lambda$, and use the definition of the Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

   (b) The exponential distribution is sometimes used to model survival times: Suppose X is the time until failure of an electronic component and let $t > 0$ and $\delta > 0$.

      i. Find $P(X > t)$, which is the probability that the component lasts t time units.
      ii. Find $P(X > t + \delta | X > t)$, which is the conditional probability that the component lasts another δ time units, given that it has already lasted t time units.
      iii. Give an interpretation to the fact that $P(X > t + \delta | X > t)$ does not depend on t. *Hint: You may look up the phrase "memorylessness".*

   (c) Assume $\lambda = 10$ and X is the time until failure (years) of an electronic component.

      i. Give the probability that the component lasts 10 years.
      ii. Given that the component has lasted 10 years, give the probability that it lasts another 10 years.

4. Let $Y \sim \text{Gamma}(\alpha, \beta)$.

   (a) Show that

   $$\mathbb{E}Y^{-k} = \frac{1}{(\alpha - 1) \cdots (\alpha - k)\beta^k}$$

   for any integer k such that $0 < k < \alpha$.

   (b) Find $\operatorname{Var}[1/Y]$.

5. Suppose you are used to passing by, on average, 7 turtles on your one-mile walking route. You wish to come up with a pmf for the number of turtles Y that you will pass by on future walks on this route. Suppose you assume that with each 5-foot stride you will pass by a turtle with probability $7/1056$ and that the event of your passing by a turtle in one stride is independent of the event of your passing by a turtle in any other stride. *Recall: 1 mile = 5280 ft* 🇺🇸 😂.

   (a) Under these assumptions, what is the probability distribution of Y?

   (b) Under these assumptions, give $\mathbb{E}Y$.

   (c) Under these assumptions, give $P(Y \leq 3)$.

   (d) Now compute $P(Y \leq 3)$ treating Y as though it were a $\text{Poisson}(\lambda)$ rv with $\lambda = 7$.

(e) Explain what you think is the point of this question.

6. Suppose you will draw 3 marbles without replacement from each of four bags. Bag 1 has 3 red marbles and 6 black marbles; bag 2 has 10 red marbles and 20 black marbles; bag 3 has 500 red marbles and 1000 black marbles; and bag 4 has an infinite number of marbles, but twice as many black marbles as red. Let X_1, X_2, and X_3 be the numbers of red marbles drawn from bag 1, bag 2, and bag 3, respectively, and let Y be the number of red marbles drawn from bag 4.

   (a) What is the probability distribution of Y?

   (b) Make the following table and fill in all the probabilities:

x	0	1	2	3
$P(X_1 = x)$				
$P(X_2 = x)$				
$P(X_3 = x)$				
$P(Y = x)$				

   (c) Explain what you think is the point of this question.

7. Let the rv X have the empirical distribution based on the data points

$$(x_1, \dots, x_5) = (1.2, 0.5, 2.3, 4, 1.6).$$

   (a) Give the pmf p_X of X.

   (b) Make an accurate drawing of the cdf F_X of X.

   (c) Compute $\mathbb{E}X$.

   (d) Compute $\operatorname{Var} X$.

8. Suppose X is the time it takes, in minutes, for you to ride your bike to work, and assume that $X \sim \text{Normal}(\mu = 15, \sigma^2 = 4)$. *Hint: Make use of the* `pnorm()` *function in R. You can't compute integrals over the Normal pdf by hand!*

   (a) Find $P(X \leq 15)$.

   (b) Find $P(X > 14)$.

   (c) Find $P(|X - 15| < 4)$.

   (d) Find an interval within which X will fall with probability 90%.

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.

- 3.12, 3.24, 3.30
- 4.6, 4.7, 4.8, 4.11, 4.13, 4.16
- 4.43, 4.48, 4.58, 4.62, 4.71, 4.79, 4.80
- 4.88, 4.89, 4.90, 4.91, 4.92, 4.94, 4.96, 4.97, 4.98, 4.106, 4.107, 4.111
- 4.114, 4.124, 4.125, 4.131