STAT 511 su 2020 hw 5

common pdfs, pmfs, expected value, variance

1. Let X have cdf given by

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b < x, \end{cases}$$

for some a and b with $a < b$.

(a) Write down the pdf of X.

We have

$$f_X(x) = \begin{cases} 0, & x < a \\ \dfrac{1}{b-a}, & a \leq x \leq b \\ 0, & b < x \end{cases}$$

(b) Draw a picture of the pdf of X.

(c) Give the support of X.

The support of X is the interval (a, b). Could also consider it to be $[a, b]$, $[a, b)$, or $(a, b]$. It is immaterial, since the random variable is continuous and therefore takes the values a and b with probability zero.

(d) Show that $\mathbb{E}X = (a + b)/2$.

We have

$$\begin{aligned} \mathbb{E}X &= \int_a^b x \cdot \frac{1}{b-a} dx \\ &= \frac{1}{b-a} \frac{b^2 - a^2}{2} \\ &= \frac{(b+a)(b-a)}{2(b-a)} \\ &= \frac{a+b}{2} \end{aligned}$$

(e) Show* that $\text{Var}\, X = (a - b)^2/12$.

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*A useful fact: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ for any x, y.

First we obtain

$$\begin{aligned}\mathbb{E}X^2 &= \int_a^b x^2 \cdot \frac{1}{b-a}dx \\ &= \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] \\ &= \frac{1}{3(b-a)}\left[(b-a)(b^2+ba+a^2)\right] \\ &= \frac{1}{3}(b^2+ba+a^2)\end{aligned}$$

So we have

$$\begin{aligned}\operatorname{Var} X &= \frac{b^2+ba+a^2}{3} - \left[\frac{a+b}{2}\right]^2 \\ &= \frac{4(b^2+ba+a^2)-3(a+b)^2}{12} \\ &= \frac{4((a+b)^2-ab)-3(a+b)^2}{12} \\ &= \frac{(a+b)^2-4ab}{12} \\ &= \frac{(a-b)^2}{12}.\end{aligned}$$

(f) Find $P(c < X < d)$ for any $c, d \in (a, b)$ with $c < d$.

We have

$$P(c < X < d) = F_X(d) - F_X(c) = \frac{d-a}{b-a} - \frac{c-a}{b-a} = \frac{d-c}{b-a}.$$

(g) Let $Y = (X-a)/(b-a)$ and find

i. $\mathbb{E}Y$.

We have

$$\mathbb{E}Y = \frac{1}{b-a}\mathbb{E}X - \frac{a}{b-a} = \frac{1}{b-a}\frac{a+b}{2} - \frac{a}{b-a} = \frac{a+b-2a}{2(b-a)} = \frac{1}{2}.$$

ii. $\operatorname{Var} Y$.

We have

$$\operatorname{Var} Y = \frac{1}{(b-a)^2} \operatorname{Var} X = \frac{1}{(b-a)^2} \frac{(b-a)^2}{12} = \frac{1}{12}.$$

(h) Find $P(1/4 < Y < 3/4)$.

We have

$$\begin{aligned} P(1/4 < Y < 3/4) &= P(1/4 < (X-a)/(b-a) < 3/4) \\ &= P(a + 1/4(b-a) < X < a + 3/4(b-a)) \\ &= \frac{a + 3/4(b-a) - a}{b-a} - \frac{a + 1/4(b-a) - a}{b-a} \\ &= 3/4 - 1/4 \\ &= 1/2. \end{aligned}$$

2. Let $X \sim \text{Beta}(\alpha, \beta)$.

(a) Show that

$$\mathbb{E}X^2 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}.$$

We have

$$\begin{aligned} \mathbb{E}X^2 &= \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx \\ &= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+2+\beta)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+2+\beta)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{(\alpha+2)-1}(1-x)^{\beta-1} dx}_{\text{integral over Gamma}(\alpha+2,\beta)\text{ pdf}} \\ &= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+2+\beta)} \\ &= \frac{\Gamma(\alpha+\beta)(\alpha+1)\alpha}{\Gamma(\alpha)(\alpha+1+\beta)(\alpha+\beta)\Gamma(\alpha+\beta)} \quad (\text{bc } \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \text{ for all } \alpha > 0) \\ &= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}. \end{aligned}$$

(b) Use the fact that $\mathbb{E}X = \alpha/(\alpha+\beta)$ and your answer to the previous part to show that

$$\operatorname{Var} X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

By the useful expression $\text{Var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2$ we have

$$\begin{aligned}\text{Var}\, X &= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left[\frac{\alpha}{\alpha+\beta}\right]^2 \\ &= \frac{(\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} \\ &= \frac{\alpha[\alpha(\alpha+\beta) + (\alpha+\beta) - \alpha(\alpha+\beta) - \alpha]}{(\alpha+\beta+1)(\alpha+\beta)^2} \\ &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.\end{aligned}$$

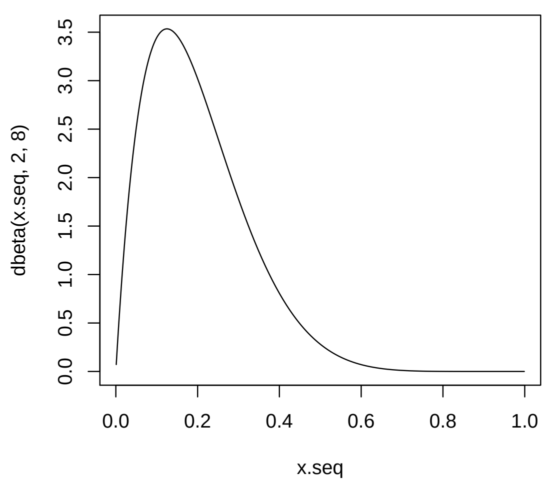
(c) Suppose X is the proportion of free-throws made over the lifetime of a randomly sampled kid, and assume that $X \sim \text{Beta}(2, 8)$.

i. Draw a picture of the pdf of X (you may use software if you wish).

The R code

```
x.seq <- seq(.001,.999,length=999)
plot(dbeta(x.seq,2,8)~x.seq,type="l")
```

produces the plot below.



ii. Compute the probability that the randomly selected kid has made fewer than 0.10 of all free throws. *Hint: Use the R function* `pbeta()`. *Access documention with* `?pbeta`.

We have $P(X < 0.10) = \texttt{pbeta(0.10,2,8)} = 0.225159$.

iii. If you were to sample 100 kids and compute the average of their lifetime free-throw success proportions, to what number would you expect this average to be close?

It should be close to $\mathbb{E}X = 2/(2+8) = 1/5$.

(d) Let $\alpha = 1$ and $\beta = 1$. For some constants a and b with $a < b$, let $Y = (b-a)X + a$. Find

i. the support of the new random variable Y.

The new random variable Y can take values in the interval (a, b).

ii. $\mathbb{E}Y$.

We have $\mathbb{E}Y = (b-a)\mathbb{E}X = (b-a)/2$.

iii. $\operatorname{Var} Y$.

We have

$$\operatorname{Var} Y = (b-a)^2 \operatorname{Var} X = (b-a)^2 \frac{1}{(1+1+1)(1+1)^2} = \frac{(b-a)^2}{12}.$$

(e) Give the pdf of X under $\alpha = 1$ and $\beta = 1$.

For $\alpha = 1$ and $\beta = 1$ we have

$$f_X(x) = \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)} x^{1-1}(1-x)^{1-1} = 1 \quad \text{for } 0 < x < 1.$$

(f) Give the cdf of X under $\alpha = 1$ and $\beta = 1$.

The cdf is that of the $\text{Uniform}(0,1)$ distribution, which is given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 < x. \end{cases}$$

(g) What is the relationship between the beta distribution and the uniform distribution?

When $\alpha = 1$ and $\beta = 1$, the Beta(α, β) distribution becomes the Uniform$(0, 1)$ distribution.

3. Let $X \sim$ Exponential(λ) for some $\lambda > 0$.
   (a) Show that for any integer $k > 0$

$$\mathbb{E}X^k = \lambda^k k!$$

   Hint: Set up the integral, do a change of variable $u = x/\lambda$, and use the definition of the Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

   We have

$$\begin{aligned}
\mathbb{E}X^k &= \int_0^\infty x^k \cdot \frac{1}{\lambda} e^{-x/\lambda} dx \\
&= \int_0^\infty (\lambda u)^k \cdot \frac{1}{\lambda} e^{-u} \lambda du \quad \text{(change of variable } u = x/\lambda) \\
&= \lambda^k \int_0^\infty u^k e^{-u} du \\
&= \lambda^k \Gamma(k+1) \\
&= \lambda^k k!.
\end{aligned}$$

   (b) The exponential distribution is sometimes used to model survival times: Suppose X is the time until failure of an electronic component and let $t > 0$ and $\delta > 0$.
      i. Find $P(X > t)$, which is the probability that the component lasts t time units.

      We have

$$P(X > t) = 1 - [1 - e^{-t/\lambda}] = e^{-t/\lambda}.$$

      ii. Find $P(X > t + \delta | X > t)$, which is the conditional probability that the component lasts another δ time units, given that it has already lasted t time units.

      We have

$$\begin{aligned}
P(X > t + \delta | X > t) &= \frac{P(\{X > t + \delta\} \cap \{X > t\})}{P(X > t)} \\
&= \frac{P(X > t + \delta)}{P(X > t)} \\
&= \frac{e^{-(t+\delta)/\lambda}}{e^{-t/\lambda}} \\
&= e^{-\delta/\lambda}.
\end{aligned}$$

iii. Give an interpretation to the fact that $P(X > t + \delta | X > t)$ does not depend on t. *Hint: You may look up the phrase "memorylessness".*

That the conditional probability $P(X > t+\delta | X > t)$ does not depend on t means that, as long as the component has not failed yet, its time until failure is like that of a brand new component. This may or may not be a good assumption; but it is a property of the exponential distribution.

(c) Assume $\lambda = 10$ and X is the time until failure (years) of an electronic component.

i. Give the probability that the component lasts 10 years.

We have

$$P(X > 10) = 1 - [1 - e^{-10/10}] = e^{-1} = 0.3678794.$$

ii. Given that the component has lasted 10 years, give the probability that it lasts another 10 years.

Due to the memoryless property, we have

$$P(X > 10 + 10 | X > 10) = P(X > 10) = 0.3678794.$$

4. Let $Y \sim \text{Gamma}(\alpha, \beta)$.

(a) Show that

$$\mathbb{E}Y^{-k} = \frac{1}{(\alpha - 1) \cdots (\alpha - k)\beta^k}$$

for any integer k such that $0 < k < \alpha$.

We have

$$\begin{aligned}
\mathbb{E}Y^{-k} &= \int_0^\infty y^{-k} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy \\
&= \frac{\Gamma(\alpha - k)\beta^{\alpha-k}}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha - k)\beta^{\alpha-k}} y^{(\alpha-k)-1} e^{-y/\beta} dy}_{\text{integral over Gamma}(\alpha - k, \beta) \text{ pdf}} \\
&= \frac{\Gamma(\alpha - k)}{(\alpha - 1) \cdots (\alpha - k)\Gamma(\alpha - k)\beta^k} \quad (\text{because } \Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \text{ for all } \alpha > 0) \\
&= \frac{1}{(\alpha - 1) \cdots (\alpha - k)\beta^k}
\end{aligned}$$

(b) Find $\text{Var}[1/Y]$.

We have

$$\begin{aligned}\text{Var}[1/Y] &= \mathbb{E}[1/Y^2] - (\mathbb{E}[1/Y])^2 \\ &= \frac{1}{(\alpha-1)(\alpha-2)\beta^2} - \left[\frac{1}{(\alpha-1)\beta}\right]^2 \\ &= \frac{(\alpha-1)-(\alpha-2)}{(\alpha-1)^2(\alpha-2)\beta^2} \\ &= \frac{1}{(\alpha-1)^2(\alpha-2)\beta^2}.\end{aligned}$$

5. Suppose you are used to passing by, on average, 7 turtles on your one-mile walking route. You wish to come up with a pmf for the number of turtles Y that you will pass by on future walks on this route. Suppose you assume that with each 5-foot stride you will pass by a turtle with probability $7/1056$ and that the event of your passing by a turtle in one stride is independent of the event of your passing by a turtle in any other stride. *Recall: 1 mile = 5280 ft* 🇺🇸 😂.

(a) Under these assumptions, what is the probability distribution of Y?

$Y \sim \text{Binomial}(1056, 7/1056)$.

(b) Under these assumptions, give $\mathbb{E}Y$.

We have $\mathbb{E}Y = 1056 \cdot (7/1056) = 7$.

(c) Under these assumptions, give $P(Y \leq 3)$.

Using the binomial cdf function in R, we have `pbinom(3,1056,7/1056)` $= 0.08107379$.

(d) Now compute $P(Y \leq 3)$ treating Y as though it were a $\text{Poisson}(\lambda)$ rv with $\lambda = 7$.

Using the Poisson cdf function in R, we have `ppois(3,7)` $= 0.08176542$.

(e) Explain what you think is the point of this question.

This question illustrates the construction of a Poisson random variable as the number of successes in an increasing number of independent Bernoulli trials, where the success probability of the trials diminishes at such a rate as to hold constant the expected total number of successes over all the trials. Letting the number of trials tend to infinity, we obtain a Poisson random variable.

6. Suppose you will draw 3 marbles without replacement from each of four bags. Bag 1 has 3 red marbles and 6 black marbles; bag 2 has 10 red marbles and 20 black marbles; bag 3 has 500 red marbles and 1000 black marbles; and bag 4 has an infinite number of marbles, but twice as many black marbles as red. Let X_1, X_2, and X_3 be the numbers of red marbles drawn from bag 1, bag 2, and bag 3, respectively, and let Y be the number of red marbles drawn from bag 4.

(a) What is the probability distribution of Y?

We have $Y \sim \text{Binomial}(3, 1/3)$.

(b) Make the following table and fill in all the probabilities:

x	0	1	2	3
$P(X_1 = x)$				
$P(X_2 = x)$				
$P(X_3 = x)$				
$P(Y = x)$				

We obtain the following table:

x	0	1	2	3
$P(X_1 = x)$	0.23809524	0.53571429	0.21428571	0.01190476
$P(X_2 = x)$	0.28078818	0.46798030	0.22167488	0.02955665
$P(X_3 = x)$	0.29599974	0.44488938	0.22222202	0.03688886
$P(Y = x)$	0.29629630	0.44444444	0.22222222	0.03703704

An example calculation is

$$P(X_2 = 2) = \frac{\binom{10}{2}\binom{20}{1}}{\binom{30}{3}} = 0.22167488.$$

(c) Explain what you think is the point of this question.

The point is to illustrate the relationship between the hypergeometric distribution and the binomial distribution. The exercise illustrates that if we are sampling without replacement from a very large population, each draw changes the composition of the population only slightly, so that the difference between sampling with and without replacement is negligible.

7. Let the rv X have the empirical distribution based on the data points

$$(x_1, \ldots, x_5) = (1.2, 0.5, 2.3, 4, 1.6).$$

(a) Give the pmf p_X of X.

We have
$$p_X(x) = \frac{1}{5} \cdot \mathbf{1}(x \in \{1.2, 0.5, 2.3, 4, 1.6\}).$$

(b) Make an accurate drawing of the cdf F_X of X.

(c) Compute $\mathbb{E}X$.

$\mathbb{E}X = (1.2 + 0.5 + 2.3 + 4 + 1.6)/5 = 1.92$.

(d) Compute $\text{Var}\, X$.

$\text{Var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = (1.2^2 + 0.5^2 + 2.3^2 + 4^2 + 1.6^2)/5 - (1.92)^2 = 1.4216$.

8. Suppose X is the time it takes, in minutes, for you to ride your bike to work, and assume that $X \sim \text{Normal}(\mu = 15, \sigma^2 = 4)$. *Hint: Make use of the* `pnorm()` *function in R. You can't compute integrals over the Normal pdf by hand!*

(a) Find $P(X \leq 15)$.

$P(X \leq 15) = 1/2$.

(b) Find $P(X > 14)$.

We have
$$P(X > 14) = P((X - 15)/2 > (14 - 15)/2) = P(Z > -1/2), \quad Z \sim \text{Normal}(0, 1),$$
and $P(Z > -1/2) = 1 - P(Z < -1/2) = 1 -$ `pnorm(-1/2)` $= 0.6914625$.

(c) Find $P(|X - 15| < 4)$.

We have
$$P(|X - 15| < 4) = P(-4 < X - 15 < 4) = P(-2 < Z < 2), \quad Z \sim \text{Normal}(0, 1),$$
and $P(-2 < Z < 2) =$ `pnorm(2) - pnorm(-2)` $= 0.9544997$.

(d) Find an interval within which X will fall with probability 90%.

Defining z_ξ as the value which satisfies $P(Z > z_\xi) = \xi$ for all $\xi \in (0, 1)$, where $Z \sim \text{Normal}(0, 1)$, we have
$$P(-z_{0.05} < Z < z_{0.05}) = 0.90,$$

where $z_{0.05} = \texttt{qnorm(.95)} = 1.644854$. So

$$\begin{aligned} 0.90 &= P(-1.644854 < Z < 1.644854) \\ &= P(2(-1.644854) + 15 < 2Z + 15 < 2(1.644854) + 15) \\ &= P(11.71029 < X < 18.28971). \end{aligned}$$

Therefore an interval in which X will fall with probability 90% is $(11.71029, 18.28971)$.

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.

- 3.12, 3.24, 3.30
- 4.6, 4.7, 4.8, 4.11, 4.13, 4.16
- 4.43, 4.48, 4.58, 4.62, 4.71, 4.79, 4.80
- 4.88, 4.89, 4.90, 4.91, 4.92, 4.94, 4.96, 4.97, 4.98, 4.106, 4.107, 4.111
- 4.114, 4.124, 4.125, 4.131