STAT 511 su 2020 hw 6

mgfs, quantiles

1. Let $X \sim \text{Poisson}(\lambda)$.
   (a) Show that the mgf of X is given by $M_X(t) = e^{\lambda(e^t - 1)}$.
   (b) Use the mgf to find
      i. $\mathbb{E}X$.
      ii. $\mathbb{E}X^2$.
      iii. $\text{Var}\,X$.
   (c) Let $Y = 3X + 1$. Give the mgf of Y and state whether Y has a Poisson distribution.

2. Let $X \sim \text{Uniform}(0, \theta)$ distribution.
   (a) Show that the mgf of X can be written as

$$1 + \frac{t\theta}{2} + \frac{(t\theta)^2}{3!} + \frac{(t\theta)^3}{4!} + \frac{(t\theta)^4}{5!} + \dots$$

   Hint: Make use of the series representation

$$e^a = \begin{cases} \sum_{i=0}^{\infty} a^i/i!, & a \neq 0 \\ 1, & a = 0. \end{cases}$$

   (b) Identify the distribution of the rv $Y = X/\theta$ by finding its mgf.

3. Let $X \sim \text{Gamma}(2, 2)$. *Hint: Make use of the* `pgamma()` *and* `qgamma()` *functions in R.*
   (a) Give $P(X > 2)$.
   (b) Give the median of X.
   (c) Find the mgf of the rv $Y = 2X - 4$ and state whether Y has a Gamma distribution.
   (d) Find $P(Y < 1)$.
   (e) Find the mgf of the rv $W = 2X$ and state whether W has a Gamma distribution.
   (f) Find $P(1 < W < 2)$.

4. Find the quantile function $Q_X(\theta) : (0, 1) \to \mathcal{X}$ for each of the following random variables (*Hint*: Set up the equation $F_X(q) = \theta$ and solve for q):
   (a) $X \sim \text{Exponential}(\lambda)$.
   (b) X having cdf given by

$$F_X(x) = \frac{1}{[1 + e^{-\tau(x-\mu)}]^{1/\nu}}, \quad -\infty < x < \infty.$$

   for some $\tau > 0$, $\nu > 0$, and $\mu \in \mathbb{R}$.

5. Consider the set of data points

$$0.27 \quad -0.63 \quad 0.87 \quad 1.73 \quad 0.02 \quad 0.37 \quad -1.31 \quad 0.74 \quad 0.04 \quad -1.05.$$

   (a) Find the θ-quantile of the empirical distribution of these data points for $\theta = (i - 0.5)/10$, for $i = 1, \ldots, 10$.
   (b) Give the θ-quantile for $\theta = (i - 0.5)/10$, for $i = 1, \ldots, 10$ of the $\text{Normal}(0, 1)$ distribution.
   (c) Make a plot of the empirical distribution quantiles (on the vertical axis) versus the $\text{Normal}(0, 1)$ quantiles. *Use whatever software you want. Print the plot or take a picture of it on your screen.*
   (d) You should see that the points fall roughly along a straight line. What is your interpretation of this?

6. Let $X \sim \text{Binomial}(3, 1/2)$.
   (a) Make a drawing of the cdf of X.
   (b) Find the θ-quantile of X for all the values $\theta = 2/16, 3/16, 8/16, 9/16, 15/16$.

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.*:

- 3.145, 3.146, 3.147, 3.148, 3.149, 3.150, 3.153, 3.154
- 4.42, 4.61
- 4.136, 4.139, 4.144, 4.145

*Ignore all references in the textbook to applets and just use R to compute probabilities that cannot be computed by hand.