STAT 511 su 2020 hw 7

joint, marginal, and conditional pdfs and pmfs

1. Let (X, Y) be a pair of random variables with joint pdf given by

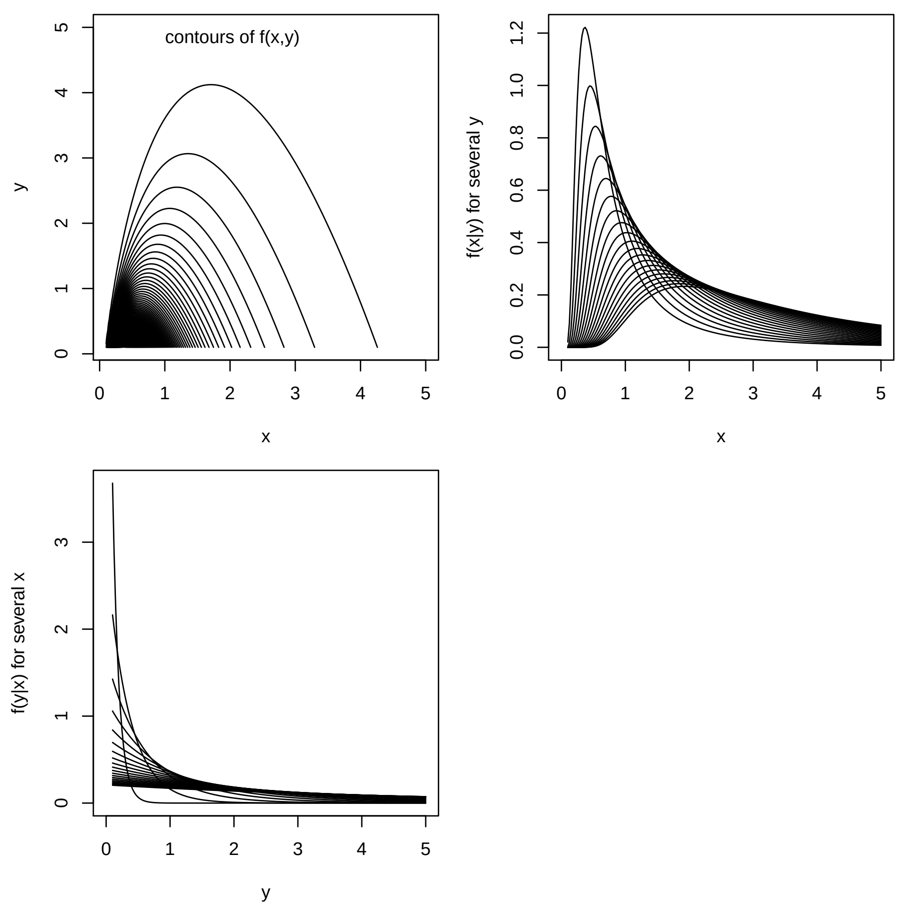
$$f(x, y) = \mathbf{1}(0 < x < 1, 0 < y < 1).$$

   (a) Find $P(X + Y \leq 1)$.
   (b) Find $P(|X - Y| \leq 1/2)$.
   (c) Find the joint cdf $F(x, y)$ of (X, Y) for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.
   (d) Find the marginal pdf f_X of X.
   (e) Find the marginal pdf f_Y of Y.
   (f) Find the conditional pdf $f(x|y)$ of $X|Y = y$ for $0 < y < 1$.

2. Let (X, Y) be a pair of random variables with joint pdf given by

$$f(x, y) = \frac{1}{x^3} e^{-1/x} e^{-y/x} \cdot \mathbf{1}(x > 0, y > 0).$$

The plots below show contours of the joint pdf $f(x, y)$ as well as the conditional densities $f(x|y)$ and $f(y|x)$ for several values of the conditioning variable.



(a) Find $P(X > Y)$.

(b) Find $E[Y/X]$.

(c) Find the marginal pdf f_X of X.

(d) Find the marginal pdf f_Y of Y.

(e) Find the conditional pdf $f(y|x)$ of $Y|X = x$ for $x > 0$.

(f) Find the conditional pdf $f(x|y)$ of $X|Y = y$ for $y > 0$.

3. Let (X, Y) be a pair of random variables with joint pdf given by

$$f(x, y) = \frac{x}{\theta} e^{-x/\theta} \mathbf{1}(0 < y < 1/x, x > 0).$$

   (a) Find $P(1 \leq X \leq 2, Y \leq 1)$.

   (b) Find the marginal pdf f_X of X.

   (c) Find $\mathbb{E}X$.

   (d) Find the marginal pdf f_Y of Y and draw a picture of it when $\theta = 1$ (you may use software). *Hint: You will have to do integration by parts.*

   (e) Give the conditional pdf $f(x|y)$ of $X|Y = y$ for $y = 1$ when $\theta = 1$.

   (f) Give the conditional pdf $f(y|x)$ of $Y|X = x$ for $x > 0$.

4. Consider rolling two dice and let (X, Y) be the random variable pair defined such that X is the sum of the rolls and Y is the maximum of the rolls (refer to Lec 11 notes).

   Find the following:

   (a) $\mathbb{E}[X/Y]$

   (b) $P(X > Y)$

   (c) $P(X = 7)$

   (d) $P(Y \leq 4)$

   (e) $P(X = 7, Y = 4)$

5. This exercise is an example of what is called Monte Carlo simulation. Sometimes it is cumbersome to compute a probability or an expected value, so we use a computer to virtually draw a large number of realizations (values of a random variable) from a distribution and we use the output to approximate the probabilities or expectations we are interested in.

   Use R to virtually toss two dice 5000 times and use the output to approximate the following quantities (turn in your R code—see example code below—and the numbers you get):

   (a) $\mathbb{E}[X/Y]$

   (b) $P(X > Y)$

   (c) $P(X = 7)$

   (d) $P(Y \leq 4)$

(e) $P(X = 7, Y = 4)$

(f) $\mathbb{E}[Y/X]$

(g) $\mathbb{E}[Y^2/X]$

Hint: Use (a)–(e) to check your answers to (a)–(e) of the previous question; your Monte Carlo results should be close to your theoretical results. Use the R code below as a guide.

```
# generate 5000 rolls of a die
roll1 <- sample(1:6,5000,replace=TRUE)
roll2 <- sample(1:6,5000,replace=TRUE)

# combine them to form a matrix with two columns,
# one column for each roll
rolls <- cbind(roll1,roll2)

# take the maximum of each row in the matrix 'rolls' and store these in Y
Y <- apply(rolls,1,max)
# take the sum of each row in the matrix 'rolls' and store these in X
X <- apply(rolls,1,sum)

# compute the average of the ratio X over Y:
mean(X/Y)

# compute the proportion of times X = 7  and Y = 4
mean( (X == 7) & (Y == 4) )
```

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 5.5, 5.6, 5.8
- 5.2, 5.16, 5.17, 5.18
- 5.20, 5.24, 5.26, 5.36, 5.38
- 5.45, 5.49, 5.51(a), 5.52, 5.61, 5.63
- 5.81
- 5.89, 5.92
- 5.112