STAT 511 su 2020 hw 7

joint, marginal, and conditional pdfs and pmfs

1. Let (X, Y) be a pair of random variables with joint pdf given by

$$f(x,y) = \mathbf{1}(0 < x < 1, 0 < y < 1).$$

(a) Find $P(X + Y \leq 1)$.

We have

$$\begin{aligned} P(X+Y \leq 1) &= \int_0^1 \int_0^{1-x} 1 \cdot dy dx \\ &= \int_0^1 (1-x) dx \\ &= \left[x - \frac{x^2}{2} \right] \Big|_0^1 \\ &= 1/2. \end{aligned}$$

(b) Find $P(|X - Y| \leq 1/2)$.

We can find this probability by setting up the corresponding double integral; however, noting that the joint pdf has a constant height of 1 over the entire joint support $(0,1) \times (0,1)$, we see can we can compute $P(|X - Y| \leq 1/2)$ by simply computing the area of the set

$$\{(x,y) : |x - y| \leq 1/2, 0 < x < 1, 0 < y < 1\}.$$

If we draw a picture, we can see that the area of this set is equal to $3/4$.

(c) Find the joint cdf $F(x, y)$ of (X, Y) for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

We have

$$F(x,y) = \begin{cases} 1, & x \geq 1, y \geq 1 \\ xy, & 0 < x < 1, 0 < y < 1 \\ x, & 0 < x < 1, y \geq 1 \\ y, & x \geq 1, 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(d) Find the marginal pdf f_X of X.

We have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} 1 \cdot \mathbf{1}(0 < x < 1, 0 < y < 1) dy \\ &= \int_0^1 1 \cdot dy \mathbf{1}(0 < x < 1) \\ &= \mathbf{1}(0 < x < 1), \end{aligned}$$

so that $X \sim \text{Uniform}(0, 1)$.

(e) Find the marginal pdf f_Y of Y.

We find also $f_Y(y) = \mathbf{1}(0 < y < 1)$, so that $Y \sim \text{Uniform}(0, 1)$.

(f) Find the conditional pdf $f(x|y)$ of $X|Y = y$ for $0 < y < 1$.

For $0 < y < 1$ and $0 < x < 1$ we have

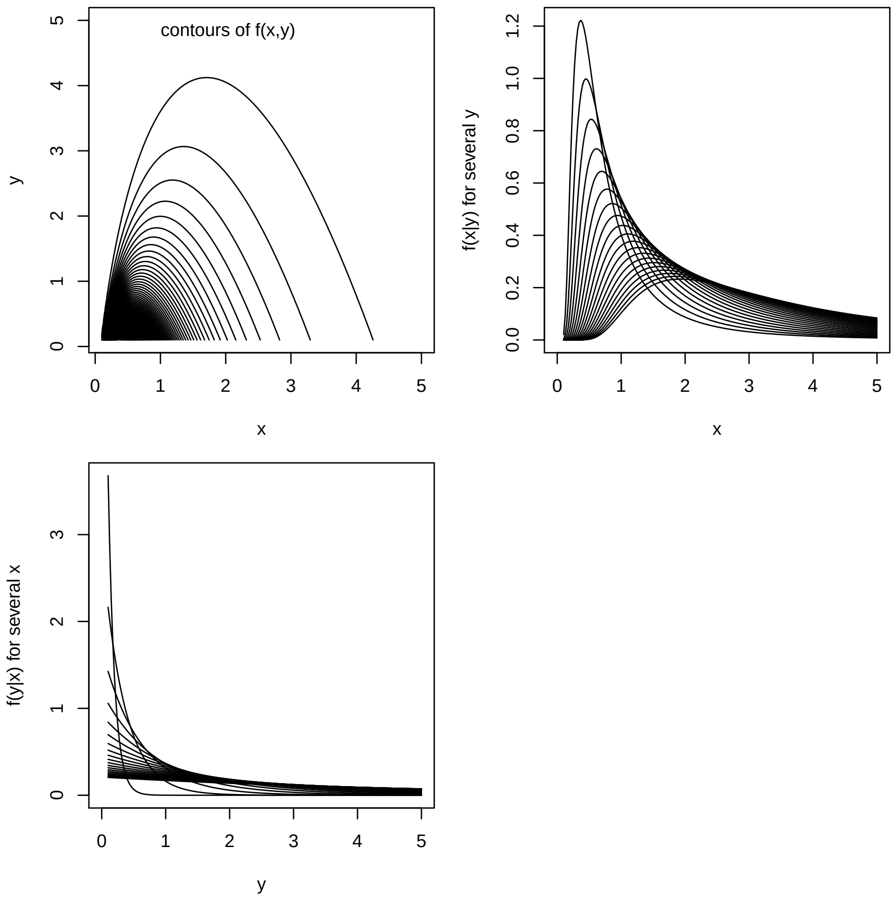
$$f(x|y) = \frac{1}{1},$$

so that $X|Y = y \sim \text{Uniform}(0, 1)$ for all $0 < y < 1$.

2. Let (X, Y) be a pair of random variables with joint pdf given by

$$f(x, y) = \frac{1}{x^3} e^{-1/x} e^{-y/x} \cdot \mathbf{1}(x > 0, y > 0).$$

The plots below show contours of the joint pdf $f(x, y)$ as well as the conditional densities $f(x|y)$ and $f(y|x)$ for several values of the conditioning variable.



(a) Find $P(X > Y)$.

We have

$$P(X > Y) = \int_0^\infty \int_0^x \frac{1}{x^3} e^{-1/x} e^{-y/x} dy dx$$
$$= \int_0^\infty \frac{1}{x^2} e^{-1/x} \underbrace{\int_0^x \frac{1}{x} e^{-y/x} dy}_{=1-e^{-1}} dx$$
$$= (1 - e^{-1}) \underbrace{\int_0^\infty \frac{1}{x^2} e^{-1/x} dx}_{=1, \text{ by } u\text{-sub: } u=1/x}$$
$$= 1 - e^{-1}.$$

(b) Find $E[Y/X]$.

We have

$$
\begin{aligned}
\mathbb{E}[Y/X] &= \int_0^\infty \int_0^\infty \frac{y}{x} \cdot \frac{1}{x^3} e^{-1/x} e^{-y/x} dy dx \\
&= \int_0^\infty \frac{1}{x^3} e^{-1/x} \underbrace{\int_0^\infty \frac{y}{x} e^{-y/x} dy}_{=x,\text{ mean of Exp}(x)} \, dx \\
&= \int_0^\infty \frac{1}{x^2} e^{-1/x} dx \\
&= 1. \qquad (\text{by } u\text{-sub: } u = 1/x)
\end{aligned}
$$

(c) Find the marginal pdf f_X of X.

We have

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^\infty \frac{1}{x^3} e^{-1/x} e^{-y/x} \mathbf{1}(x > 0, y > 0) dy \\
&= \frac{1}{x^2} e^{-1/x} \underbrace{\int_0^\infty \frac{1}{x} e^{-y/x} dy}_{=1,\text{ pdf of Exp}(x)} \cdot \mathbf{1}(x > 0) \\
&= \frac{1}{x^2} e^{-1/x} \cdot \mathbf{1}(x > 0).
\end{aligned}
$$

(d) Find the marginal pdf f_Y of Y.

We have

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^\infty \frac{1}{x^3} e^{-1/x} e^{-y/x} \mathbf{1}(x > 0, y > 0) dx \\
&= \int_0^\infty \frac{1}{x^3} e^{-\frac{1}{x}(1+y)} dx \cdot \mathbf{1}(y > 0) \\
&= \frac{1}{(1+y)^2} \underbrace{\int_0^\infty u e^{-u} du}_{=1} \cdot \mathbf{1}(y > 0) \qquad (u\text{-sub: } u = \frac{1}{x}(1+y)) \\
&= \frac{1}{(1+y)^2} \cdot \mathbf{1}(y > 0)
\end{aligned}
$$

(e) Find the conditional pdf $f(y|x)$ of $Y|X = x$ for $x > 0$.

For $x > 0$ and $y > 0$, we have

$$f(y|x) = \frac{\frac{1}{x^3}e^{-1/x}e^{-y/x}}{\frac{1}{x^2}e^{-1/x}} = \frac{1}{x}e^{-y/x},$$

so that $Y|X = x \sim \text{Exponential}(x)$.

(f) Find the conditional pdf $f(x|y)$ of $X|Y = y$ for $y > 0$.

For $x > 0$ and $y > 0$, we have

$$f(x|y) = \frac{\frac{1}{x^3}e^{-1/x}e^{-y/x}}{1/(1+y)^2}.$$

3. Let (X, Y) be a pair of random variables with joint pdf given by

$$f(x, y) = \frac{x}{\theta}e^{-x/\theta}\mathbf{1}(0 < y < 1/x, x > 0).$$

(a) Find $P(1 \leq X \leq 2, Y \leq 1)$.

We have

$$\begin{aligned} P(1 \leq X \leq 2, Y \leq 1) &= \int_1^2 \int_0^{1/x} \frac{x}{\theta}e^{-x/\theta}dydx \\ &= \int_1^2 \frac{1}{\theta}e^{-x/\theta}dx \\ &= e^{-1/\theta} - e^{-2/\theta}. \end{aligned}$$

(b) Find the marginal pdf f_X of X.

We have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{x}{\theta}e^{-x/\theta}\mathbf{1}(0 < y < 1/x, x > 0)dy \\ &= \int_0^{1/x} \frac{x}{\theta}e^{-x/\theta}dy\mathbf{1}(x > 0) \\ &= \frac{1}{\theta}e^{-x/\theta}\mathbf{1}(x > 0), \end{aligned}$$

so that $X \sim \text{Exponential}(\theta)$.

(c) Find $\mathbb{E}X$.

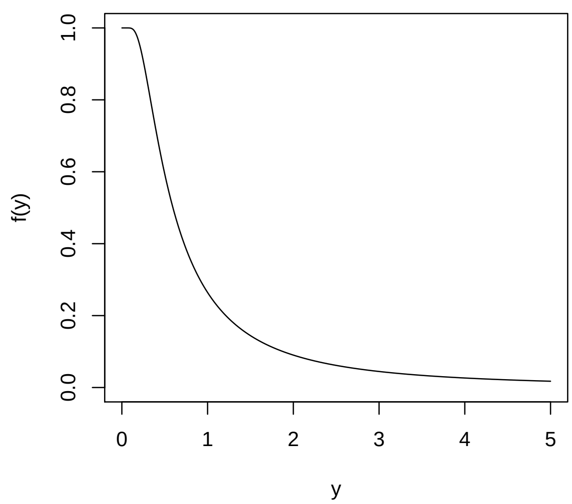
Since $X \sim \text{Exponential}(\theta)$, we have $\mathbb{E}X = \theta$.

(d) Find the marginal pdf f_Y of Y and draw a picture of it when $\theta = 1$ (you may use software). *Hint: You will have to do integration by parts.*

We have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} \frac{x}{\theta} e^{-x/\theta} \mathbf{1}(0 < y < 1/x, x > 0) dx \\ &= \int_0^{1/y} x \cdot \frac{1}{\theta} e^{-x/\theta} dx \cdot \mathbf{1}(y > 0) \\ &= \left[-xe^{-x/\theta} \Big|_0^{1/y} - \int_0^{1/y} -e^{-x/\theta} dx \right] \cdot \mathbf{1}(y > 0) \qquad \text{(by parts: } u = x,\ dv = \theta^{-1} e^{-x/\theta}) \\ &= \left[\theta - e^{-1/(y\theta)} \left(\theta + \frac{1}{y} \right) \right] \cdot \mathbf{1}(y > 0). \end{aligned}$$

With $\theta = 1$, the function looks like this:



(e) Give the conditional pdf $f(x|y)$ of $X|Y = y$ for $y = 1$ when $\theta = 1$.

For $0 < y < 1/x$ and $x > 0$ (which is the same as $0 < x < 1/y$, $y > 0$), we have

$$f(x|y) = \frac{\frac{x}{\theta} e^{-x/\theta}}{\theta - e^{-1/(y\theta)} \left(\theta + \frac{1}{y} \right)},$$

so, for $y = 1$ and $\theta = 1$, we have

$$f(x|1) = \frac{xe^{-x}}{1 - 2e^{-1}} \mathbf{1}(0 < x < 1).$$

(f) Give the conditional pdf $f(y|x)$ of $Y|X = x$ for $x > 0$.

We have

$$f(y|x) = \frac{\frac{x}{\theta} e^{-x/\theta}}{\frac{1}{\theta} e^{-x/\theta}} \mathbf{1}(0 < y < 1/x) = x \cdot \mathbf{1}(0 < y < 1/x),$$

so that $Y|X = x \sim \text{Uniform}(0, 1/x)$.

4. Consider rolling two dice and let (X, Y) be the random variable pair defined such that X is the sum of the rolls and Y is the maximum of the rolls (refer to Lec 11 notes).

Find the following:

(a) $\mathbb{E}[X/Y]$

We have $\mathbb{E}X = 57/36$.

(b) $P(X > Y)$

We have $P(X > Y) = 1$, since the sum of the rolls must always exceed the maximum of the rolls.

(c) $P(X = 7)$

We have $P(X = 7) = 1/6$.

(d) $P(Y \leq 4)$

We have$P(Y \leq 4) = 4/9$

(e) $P(X = 7, Y = 4)$

We have $P(X = 7, Y = 4) = 1/18$.

5. This exercise is an example of what is called Monte Carlo simulation. Sometimes it is cumbersome to compute a probability or an expected value, so we use a computer to virtually draw a large number of realizations (values of a random variable) from a distribution and we use the output to approximate the probabilities or expectations we are interested in.

Use R to virtually toss two dice 5000 times and use the output to approximate the following quantities (turn in your R code—see example code below—and the numbers you get):

(a) $\mathbb{E}[X/Y]$

(b) $P(X > Y)$

(c) $P(X = 7)$

(d) $P(Y \leq 4)$

(e) $P(X = 7, Y = 4)$

(f) $\mathbb{E}[Y/X]$

(g) $\mathbb{E}[Y^2/X]$

Hint: Use (a)–(e) to check your answers to (a)–(e) of the previous question; your Monte Carlo results should be close to your theoretical results. Use the R code below as a guide.

```
# generate 5000 rolls of a die
roll1 <- sample(1:6,5000,replace=TRUE)
roll2 <- sample(1:6,5000,replace=TRUE)

# combine them to form a matrix with two columns,
# one column for each roll
rolls <- cbind(roll1,roll2)

# take the maximum of each row in the matrix 'rolls' and store these in Y
Y <- apply(rolls,1,max)
# take the sum of each row in the matrix 'rolls' and store these in X
X <- apply(rolls,1,sum)

# compute the average of the ratio X over Y:
mean(X/Y)

# compute the proportion of times X = 7  and Y = 4
mean( (X == 7) & (Y == 4) )
```

Optional (do not turn in) problems for additional study from Wackerly, Mendenhall, Scheaffer, 7th Ed.:

- 5.5, 5.6, 5.8
- 5.2, 5.16, 5.17, 5.18
- 5.20, 5.24, 5.26, 5.36, 5.38
- 5.45, 5.49, 5.51(a), 5.52, 5.61, 5.63
- 5.81
- 5.89, 5.92
- 5.112