STAT 511 su 2020 hw 8

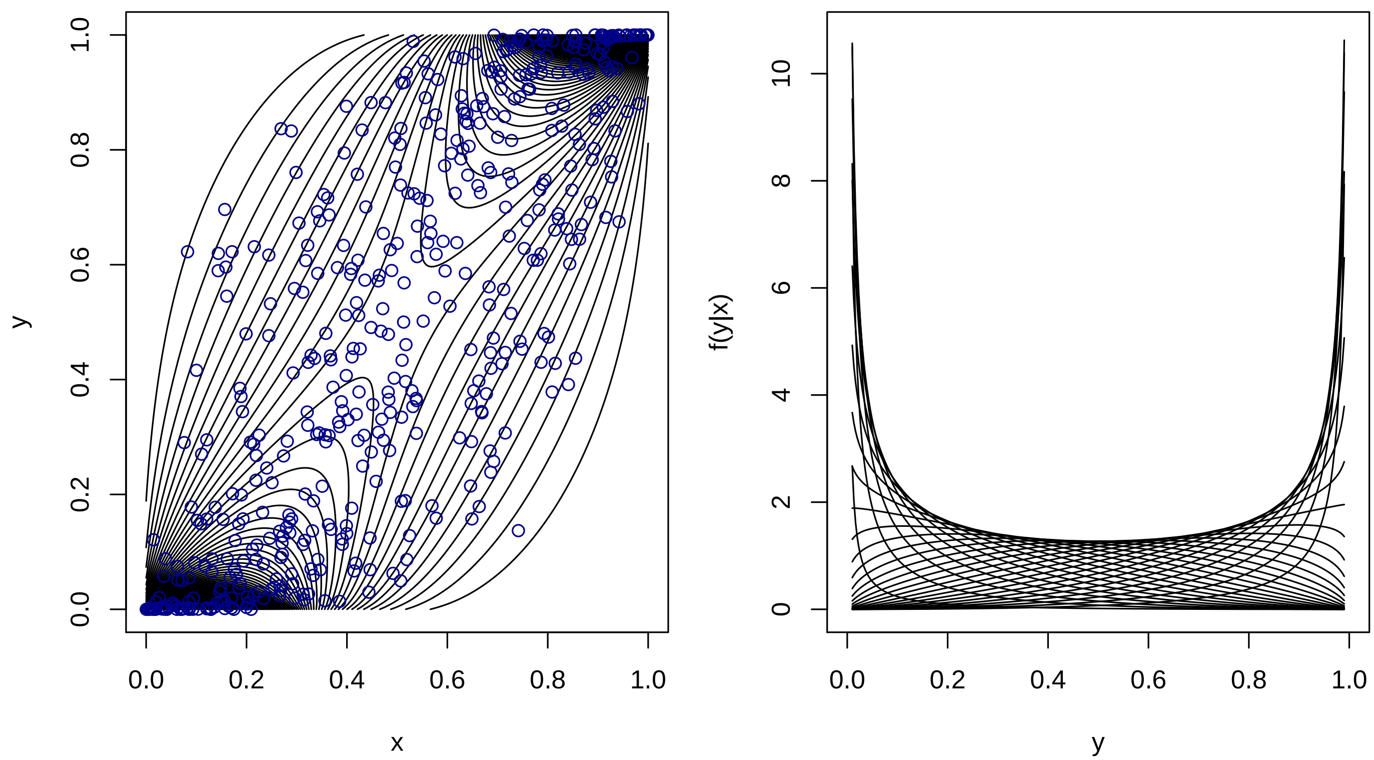
covariance, correlation, independence, hierarchical models

1. Show that for any two rvs X and Y, $\text{Cov}(X, Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$.

2. Let (X, Y) be a pair of random variables such that

$$Y|X \sim \text{Beta}(3X, 3(1-X))$$
$$X \sim \text{Uniform}(0, 1).$$

The plots below show contours of the joint density of X and Y, with 500 realizations of (X, Y) overlaid, as well as the conditional pdfs of $Y|X = x$ for several values of x.



(a) Give the joint pdf of (X, Y).

(b) Check whether X and Y are independent.

(c) Write down the integral which would yield the marginal pdf of Y.

(d) Find $\mathbb{E}[Y|X]$.

(e) Find $\mathbb{E}Y$.

(f) Find $\text{Var}[Y|X]$.

(g) Find $\text{Var}\, Y$.

(h) Use the following code to generate 50,000 realizations of (X, Y). Report the mean and the variance of the Y values (check whether these support your results for $\mathbb{E}Y$ and $\text{Var}\, Y$).

```
n <- 50000
X <- runif(n,0,1)
Y <- rbeta(n, shape1 = 3*X, shape2 = 3*(1-X))
```

3. Consider a game in which a player shoots 3 free throws; if the player makes i free throws, she draws one bill at random from a bag containing $i+1$ ten-dollar bills and $5-(i+1)$ one-dollar bills. Let X be the number of free throws she makes and Y be the amount of money she wins and assume that she makes free-throws with probability $1/2$.
   (a) Tabulate the marginal probabilities $P(X = x)$ for $x \in \mathcal{X}$.
   (b) Tabulate the joint probabilities $P(X = x, Y = y)$ for $(x, y) \in \mathcal{X} \times \mathcal{Y}$.
   (c) Tabulate the marginal probabilities $P(Y = y)$ for $y \in \mathcal{Y}$.
   (d) Check whether X and Y are independent random variables.
   (e) Compute $\mathbb{E}X$.
   (f) Compute $\mathbb{E}Y$.
   (g) Compute $\text{Cov}(X, Y)$.
   (h) Compute $\mathbb{E}[Y|X = 1]$.

4. Let (Z_1, Z_2) be a pair of rvs with the standard bivariate Normal distribution with correlation ρ, so that their joint pdf is given by

$$f(z_1, z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\frac{1}{1-\rho^2}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\right] \quad \text{for all } z_1, z_2 \in \mathbb{R}.$$

   (a) Show that Z_1 and Z_2 are independent if $\rho = 0$.
   (b) Show that the marginal pdf of Z_1 is the $\text{Normal}(0, 1)$ distribution.
   (c) Show that $Z_2|Z_1 = z_1 \sim \text{Normal}(\rho z_1, 1 - \rho^2)$.
   (d) Show that $\text{Cov}(Z_1, Z_2) = \rho$. *Hint: Obtain $\mathbb{E}Z_1Z_2$ via iterated expectation.*
   (e) Show that $\text{corr}(Z_1, Z_2) = \rho$.

5. A random spectator is to be selected from the audience of a basketball game and given the chance to shoot 10 free throws. Let Y be the number of free throws made by the selected spectator and let P be the probability with which the selected spectator makes a free throw on any attempt. Assume that P and Y follow the hierarchical model

$$\begin{aligned} Y|P &\sim \text{Binomial}(10, P) \\ P &\sim \text{Beta}(2, 2). \end{aligned}$$

   (a) Run a Monte Carlo simulation to generate many realizations of Y. Use the following R code:

```
S <- 10000
P <- rbeta(S,2,2)
Y <- rbinom(S,10,P)
```

Use the realizations of Y to get approximate values for

i. $\mathbb{E}Y$.

ii. $\operatorname{Var} Y$.

(b) Find $\mathbb{E}[Y|P]$

(c) Find $\mathbb{E}[Y]$.

(d) Find $\operatorname{Var}[Y|P]$.

(e) Find $\operatorname{Var}[Y]$.

(f) Find values of α and β such that $\mathbb{E}Y = 4$ when $P \sim \text{Beta}(\alpha, \beta)$.