STAT 511 su 2020 Exam I

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This is a take-home test due to the COVID-19 suspension of face-to-face instruction. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Consider a ten-sided die of which the sides display the numbers 1, 2, 3, and 4 according to this table:

side of die	1	2	3	4	5	6	7	8	9	10
number displayed	1	1	1	1	2	2	2	3	3	4

Rolling two such dice is an experiment with the sample space

$$\mathcal{S} = \left\{ \begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{array} \right\}$$

(a) Find the probability of the outcome $(1,1)$. *Hint: It is not one-sixteenth.*

Solution: Since we roll a 1 with probability 4/10 and the two rolls are independent, we have

$$P((1,1)) = 4/10 \cdot 4/10 = 16/100 = 4/25.$$

(b) Find the probability of not rolling doubles.

Solution: It will require less calculation to compute the probability of rolling doubles (the complement event) and subtract it from 1. We have

$$\begin{aligned} P(\text{Not rolling doubles}) &= 1 - P(\text{Rolling doubles}) \\ &= 1 - P(\{(1,1),(2,2),(3,3),(4,4)\}) \\ &= 1 - [P((1,1)) + P((2,2)) + P((3,3)) + P((4,4))] \\ &= 1 - [(4/10)^2 + (3/10)^2 + (2/10)^2 + (1/10)^2] \\ &= 1 - (16+9+4+1)/100 \\ &= 0.7. \end{aligned}$$

(c) Find the probability that the sum of the rolls is at least seven.

Solution:

$$P(\text{Sum is at least 7}) = P(\{(4,3),(3,4),(4,4)\}) = 2(1/10 \cdot 2/10) + (1/10)^2 = 5/100 = 0.05.$$

(d) Find the probability that doubles are rolled given that the sum of the rolls is less than seven.

Solution:

$$\begin{aligned} P(\text{Rolling doubles}|\text{Sum less than 7}) &= \frac{P(\text{Roll doubles} \cap \text{Sum less than 7 })}{P(\text{Sum less than 7})} \\ &= [(4/10)^2 + (3/10)^2 + (2/10)^2]/(1 - 5/100) \\ &= 29/95 \\ &= 0.3052632 \end{aligned}$$

(e) Find the probability that the sum of the rolls is less than seven given that doubles are rolled.

Solution:

$$\begin{aligned} P(\text{Sum less than 7}|\text{Rolling doubles}) &= \frac{P(\text{Roll doubles} \cap \text{Sum less than 7 })}{P(\text{Rolling doubles})} \\ &= \frac{(4/10)^2 + (3/10)^2 + (2/10)^2}{(4/10)^2 + (3/10)^2 + (2/10)^2 + (1/10)^2} \\ &= 29/30 \\ &= 0.9666667 \end{aligned}$$

3. The patio of a restaurant has three tables at which two, four, and five patrons can be seated, respectively. A party of eleven is to be seated on the patio.

(a) In how many ways can the party of eleven be seated at the three tables (it is not important in which seats at each table the patrons sit).

Solution: This is a partition of the 11 members of the party into groups of sizes 2, 4, and 5. The number of ways in which this can be done is

$$\#\text{ ways } = \frac{11!}{2!4!5!} = 6930.$$

(b) Three friends among the party wish to sit together. In how many ways can the party of eleven be seated such that the three friends sit at the same table?

Solution:

$$\#\text{ ways } = \frac{8!}{2!1!5!} + \frac{8!}{2!4!2!} = 588.$$

(c) Suppose a seating arrangement is randomly selected among all possible seating arrangements such that each arrangement is equally likely. Find the probability that the three friends are seated at the same table.

Solution:

$$P(\text{Three friends at same table}) = \frac{588}{6930} = \frac{14}{165} = 0.08484848.$$

(d) Suppose one of the three friends arrives early and seats herself at the table at which five patrons can be seated. Supposing that the remaining members of the party are assigned seats at random, find the probability that the three friends end up sitting at the same table.

Solution:

$$\begin{aligned} P(\text{Three friends at same table}|\text{One sits at biggest table}) &= \frac{8!}{2!4!2!} \div \frac{10!}{2!4!4!} \\ &= \frac{2}{15} \\ &= 0.1333333. \end{aligned}$$

(e) Find the probability that the three friends end up sitting at the same table given that the friend arriving early sits at the table at which four patrons can be seated.

Solution:

$$\begin{aligned} P(\text{Three friends at same table}|\text{One sits at medium table}) &= \frac{8!}{2!1!5!} \div \frac{10!}{2!3!5!} \\ &= \frac{1}{15} \\ &= 0.06666667. \end{aligned}$$

4. An individual randomly selected from a population is to be tested for the presence of antibodies to a virus. Define the events

$$\begin{aligned} A &= \text{the individual has antibodies} \\ + &= \text{the individual tests positive} \\ - &= \text{the individual tests negative} \end{aligned}$$

Suppose antibodies are present in 10% of the population and

$$\begin{aligned} P(+|A) &= 0.999 \quad \text{(the sensitivity of the test)} \\ P(-|A^c) &= 0.992 \quad \text{(the specificity of the test)} \end{aligned}$$

(a) Find $P(A|+)$.

Solution:

$$\begin{aligned} P(A|+) &= \frac{P(+|A)P(A)}{P(+|A)P(A)+P(+|A^c)P(A^c)} \\ &= \frac{(0.999)(0.10)}{(0.999)(0.10)+(1-0.992)(1-0.10)} \\ &= 0.9327731. \end{aligned}$$

(b) Suppose a large number of individuals are selected at random from the population and tested for antibodies. For every 1000 who test positive, how many of these can you expect to have been false positives (it is a false positive when an individual without antibodies tests positive).

Solution: We would expect $1000 \cdot (1 - 0.9327731) = 67.2269$ to have been false positives.

(c) Find $P(A^c|-)$.

Solution:

$$\begin{aligned} P(A^c|-) &= \frac{P(-|A^c)P(A^c)}{P(-|A^c)P(A^c)+P(-|A)P(A)} \\ &= \frac{(0.992)(1-0.10)}{(0.992)(1-0.10)+(1-0.999)(0.10)} \\ &= 0.999888 \end{aligned}$$

(d) Suppose an individual with antibodies is tested twice and that the outcomes of the two tests are independent. Find the probability that both tests are positive.

Solution: Denote the event of two positive tests with $++$. We have $P(++|A) = (0.999)^2 = 0.998001$.

(e) Find the probability that an individual without antibodies tests positive twice.

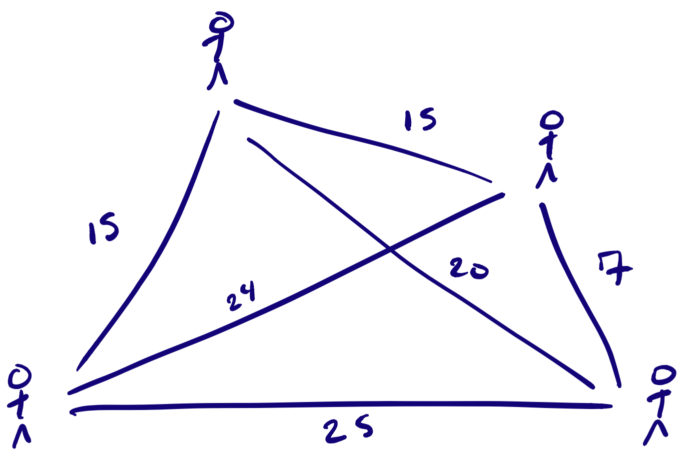
Solution: We have $P(++|A^c) = (1-0.992)^2 = 0.000064$.

(f) Find the probability that an individual who tests positive twice has antibodies to the virus.

Solution: We have

$$
\begin{aligned}
P(A|++) &= \frac{P(++|A)P(A)}{P(++|A)P(A) + P(++|A^c)P(A^c)} \\
&= \frac{(0.998001)(0.10)}{(0.998001)(0.10) + (0.000064)(0.90)} \\
&= 0.9994232.
\end{aligned}
$$

5. Four mathematically inclined buddies decide to enjoy each other's company while constraining themselves to socially distance according to the below-depicted Brahmagupta quadrilateral, with an integer-valued distance (in feet) between each pair of them:



Consider choosing two of the four friends at random and define the rv X as the distance between them.

(a) Tabulate the probability distribution of X in a table of the form

x	$\ldots$
$P_X(X = x)$	$\ldots$

Solution:

x	7	15	20	24	25
$P_X(X = x)$	1/6	2/6	1/6	1/6	1/6

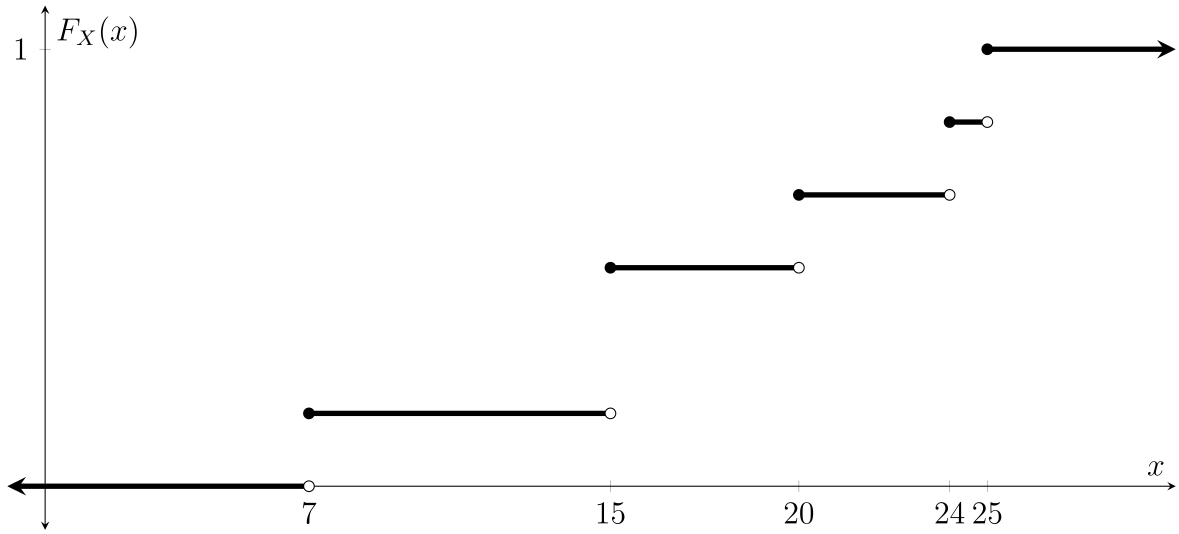
(b) Give the cdf $F_X(x)$ for all $x \in \mathbb{R}$.

Solution: We have

$$
F_X(x) = \begin{cases} 0, & x < 7 \\ 1/6, & 7 \leq x < 15 \\ 3/6, & 15 \leq x < 20 \\ 4/6, & 20 \leq x < 24 \\ 5/6, & 24 \leq x < 25 \\ 1, & 25 \leq x \end{cases}
$$

(c) Draw a detailed picture of F_X.

Solution:



(d) State whether X is discrete or continuous and explain your answer.

Solution: The rv X is discrete because the cdf is a step function.

(e) Find $P(X \leq 15)$.

Solution:

$$P(X \leq 15) = 3/6.$$

(f) Find $P(7 < X < 24)$.

Solution:

$$P(7 < X < 24) = 3/6.$$