STAT 511 su 2020 Exam II

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This is a take-home test due to the COVID-19 suspension of face-to-face instruction. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	$\operatorname{Var} X$
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$:

z	0.674	0.841	1.036	1.282	1.645	1.96	2.576
$\Phi(z)$	0.75	0.80	0.85	0.9	0.95	0.975	0.995

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. A beginning unicyclist is attempting to unicycle from his garage to the end of his driveway—a distance of 200 feet. His skill is such that, once mounted on the unicycle, he will fall within a distance of x feet from his starting point with probability given by $x/250 - x^2/500^2$, for $0 < x < 500$. Let X be the distance from the unicyclist's starting point at which he falls.

   (a) Give the cdf $F_X(x)$ of X for all $x \in \mathbb{R}$.

   Solution: We have
$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x/250 - x^2/500^2, & 0 < x < 500 \\ 1, & 500 \leq x. \end{cases}$$

   (b) Give the pdf $f_X(x)$ of X for all $x \in \mathbb{R}$.

   Solution: We have
$$f_X(x) = \frac{1}{250}\left(1 - \frac{x}{500}\right)\mathbf{1}(0 < x < 500).$$

   (c) With what probability does the unicyclist fall before reaching the end of his driveway?

   Solution: This is $P(X \leq 200) = F_X(200) = 200/250 + 200^2/500^2 = 16/25 = 0.64$.

   (d) With what probability does the unicyclist reach the end of his driveway before falling?

   Solution: This is $P(X > 200) = 1 - P(X \leq 200) = 1 - 16/25 = 9/25 = 0.36$.

   (e) What is the maximum distance the unicyclist is capable of reaching?

   Solution: For any $x \geq 500$, $F_X(x) = 1$, so with probability equal to 1, he falls before reaching 500 feet. Therefore, 500 feet is the maximum distance he is capable of reaching.

   (f) Find the median of X.

   Solution: We can find the median of X by solving $F_X(q) = 1/2$ for q. We have
$$1/2 = x/250 - x^2/500^2 \iff x^2/500^2 - x/250 + 1/2 = 0,$$
   of which the solutions, by the quadratic formula, are
$$x = 500 \pm \sqrt{2} \cdot 250.$$

Since only the solution $500 - \sqrt{2} \cdot 250$ lies in the support of X, the median of X is

$$q = 500 - \sqrt{2} \cdot 250 = 146.4466.$$

(g) Find $\mathbb{E}X$.

Solution: We have

$$\begin{aligned}\mathbb{E}X &= \int_0^{500} x \cdot \frac{1}{250}\left(1 - \frac{x}{500}\right) dx \\ &= 500/3 \\ &= 166.6667.\end{aligned}$$

(h) Find the standard deviation of X.

Solution: We have

$$\begin{aligned}\mathbb{E}X^2 &= \int_0^{500} x^2 \cdot \frac{1}{250}\left(1 - \frac{x}{500}\right) dx \\ &= 500^2/6 \\ &= 41666.67,\end{aligned}$$

so that

$$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 500^2/6 - (500/3)^2 = 500^2/18 = 13888.89.$$

Denoting by σ_X the standard deviation of X, we have

$$\sigma_X = \sqrt{\operatorname{Var} X} = 500/\sqrt{18} = 117.8511.$$

(i) Let $Y = (200 - X)/3$ represent the remaining distance, in yards, to the end of the driveway from the garage (if he passes the end of the driveway, Y will be a negative number).

i. Find $\mathbb{E}Y$.

Solution: We have

$$\mathbb{E}Y = 200/3 - (1/3)500/3 = 100/9 = 11.11111.$$

ii. Find $\operatorname{Var} Y$.

Solution: We have $\operatorname{Var} Y = (1/3)^2 \operatorname{Var} X = (1/9)500^2/18 = 500^2/162 = 1543.21.$

3. Let $X \sim \text{Gamma}(4, 5)$ and let $Y = (X - 20)/10$.

(a) Give the mgf M_X of X.

Solution: We have
$$M_X(t) = (1 - 5t)^{-4} \quad \text{for} \quad t < 1/5.$$

(b) Give $M_X^{(1)}(0)$, that is the first derivative of M_X evaluated at zero.

Solution: This is $M_X^{(1)}(0) = 20$.

(c) Give $\mathbb{E}X^2$.

Solution: We have $\mathbb{E}X^2 = \operatorname{Var} X + (\mathbb{E}X)^2 = 4 \cdot 5^2 + (4 \cdot 5)^2 = 100 + 400 = 500$.

(d) Give the mgf of Y.

Solution: We have
$$M_Y(t) = M_{X/10-2}(t) = e^{-2t} M_X(t/10) = e^{-2t}(1 - 5(t/10))^{-4} = e^{-2t}(1 - t/2)^{-4}.$$

(e) State whether Y has a Gamma distribution and explain your answer.

Solution: Since the mgf of Y is not the mgf of a Gamma distribution, Y does not have a Gamma distribution.

(f) Give $\mathbb{E}Y$.

Solution: We have $\mathbb{E}Y = \mathbb{E}[(X - 20)/10] = (1/10)\mathbb{E}X - 2/2 = 20/10 - 2 = 0$.

(g) Give $\operatorname{Var} Y$.

Solution: We have $\operatorname{Var} Y = (1/100) \operatorname{Var} X = (1/100) 4 \cdot 5^2 = 1$.

(h) Give $M_Y^{(2)}(0)$, that is the second derivative of M_Y evaluated at zero.

Solution: This is the second moment of Y, which is equal to 1 since $\mathbb{E}Y = 0$ and $\operatorname{Var} Y = 1$.

4. Suppose there is a game such that the distribution of the score X of a player following a certain strategy has the cumulative probabilities given in the table below. The possible scores are $15, 16, \ldots, 40$.

x	$P(X \leq x)$
15	0.0001
16	0.0002
17	0.0004
18	0.0019
19	0.0033
20	0.0106
21	0.0300
22	0.0686
23	0.1314
24	0.2317
25	0.3522
26	0.4955
27	0.6356
28	0.7573
29	0.8512
30	0.9115
31	0.9403
32	0.9617
33	0.9843
34	0.9897
35	0.9952
36	0.9968
37	0.9990
38	0.9996
39	0.9999
40	1.0000

(a) Give $P(X \leq 27.5)$.

Solution: This is 0.6356.

(b) Give $P(X = 27)$.

Solution: $P(X = 27) = P(X \leq 27) - P(X \leq 26) = 0.6356 - 0.4955 = 0.1401$.

(c) Give the 1st quartile of X (the 0.25 quantile).

Solution: This is the value 25.

(d) Give the 0.95 quantile of X.

Solution: This is the value 32.

5. Let $X \sim \text{Normal}(\mu = 20, \sigma^2 = 4)$. *For parts (b) and (c) use the table on the front page of the test.*

   (a) Give the mgf M_X of X.

   Solution: We have $M_X(t) = e^{20t+4t^2/2}$.

   (b) Find the 0.10 quantile of X.

   Solution: The 0.10 quantile of the standard Normal distribution is -1.282. So the 0.10 quantile of any Normal distribution lies at -1.282 standard deviations below the mean. Thus the 0.10 quantile of X is $20 - 1.282(2) = 17.436$.

   (c) Find an interval within which X lies with probability 0.60.

   Solution: A standard Normal random variable lies between -0.841 and 0.841 with probability 0.60, so any Normal random variable lies within 0.841 standard deviations of its mean with probability 0.60. Therefore X lies in the interval

   $$(20 - 0.841(2), 20 - 0.841(2)) = (18.318, 21.682)$$

   with probability 0.60.

   (d) Find the distribution of $Y = 3X - 10$ by finding the mgf M_Y of Y.

   Solution: The mgf of Y is given by

   $$M_Y(t) = M_{3X-10}(t) = e^{-10t}M_X(3t) = e^{-10t}e^{20(3t)+4(3t)^2/2} = e^{50t+36t^2/2}.$$

   Therefore $Y \sim \text{Normal}(\mu = 50, \sigma^2 = 36)$.