STAT 511 su 2020 Final Exam

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This is a take-home test due to the COVID-19 suspension of face-to-face instruction. Do not communicate with classmates about the exam until after its due date/time. You may

- *Use your notes and the lecture notes.*
- *Use books.*
- *NOT work together with others.*

Write all answers on blank sheets of paper; then take pictures and merge to a PDF. Upload a single PDF to Blackboard.

pmf/pdf	$\mathcal{X}$	$M_X(t)$	$\mathbb{E}X$	Var X
$p_X(x;p) = p^x(1-p)^{1-x}$,	$x = 0, 1$	$pe^t + (1-p)$	p	$p(1-p)$
$p_X(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x}$,	$x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	np	$np(1-p)$
$p_X(x;p) = (1-p)^{x-1}p$,	$x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	p^{-1}	$(1-p)p^{-2}$
$p_X(x;p,r) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$,	$x = r, r+1, \dots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	rp^{-1}	$r(1-p)p^{-2}$
$p_X(x;\lambda) = e^{-\lambda}\lambda^x/x!$	$x = 0, 1, \dots$	$e^{\lambda(e^t-1)}$	λ	λ
$p_X(x;N,M,K) = \binom{M}{x}\binom{N-M}{K-x}/\binom{N}{K}$	$x = 0, 1, \dots, K$	¡complicadísimo!	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-K)(N-M)}{N(N-1)}$
$p_X(x;K) = \frac{1}{K}$	$x = 1, \dots, K$	$\frac{1}{K}\sum_{x=1}^K e^{tx}$	$\frac{K+1}{2}$	$\frac{(K+1)(K-1)}{12}$
$p_X(x;x_1,\dots,x_n) = \frac{1}{n}$	$x = x_1, \dots, x_n$	$\frac{1}{n}\sum_{i=1}^n e^{tx_i}$	$\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$	$\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$
$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$e^{\mu t + \sigma^2 t^2/2}$	μ	σ^2
$f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$	$0 < x < \infty$	$(1-\beta t)^{-\alpha}$	$\alpha\beta$	$\alpha\beta^2$
$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$1 + \sum_{k=1}^\infty \frac{t^k}{k!}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

The table below gives some values of the function $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$:

z	0.674	0.841	1.036	1.282	1.645	1.96	2.576
$\Phi(z)$	0.75	0.80	0.85	0.9	0.95	0.975	0.995

1. Copy down this sentence on your answer sheet and put your signature underneath: *I have not collaborated with any other student on this exam. The work I have presented is my own.*

2. Let X and Y each have the Uniform$(0, 1)$ distribution, and suppose $\text{corr}(X, Y) = 1/2$.
   (a) Give $\text{Var}\, X$ and $\text{Var}\, Y$.
   (b) Find $\text{Cov}(X, Y)$.
   (c) Find $\text{Var}(X + Y)$.
   (d) Find $\text{Var}(X - Y)$.
   (e) Suppose $\text{corr}(X, Y) = \rho$ for some $\rho \in [-1, 1]$. State whether $\mathbb{E}(X - Y)^2$ is an increasing or a decreasing function of ρ. Justify your answer using intuition or by finding the expression for $\mathbb{E}(X - Y)^2$ in terms of ρ.

3. Let X and Y be random variables such that

$$Y|X \sim \text{Gamma}(3, 1/X)$$
$$X \sim \text{Gamma}(3, 3).$$

   (a) Write down the joint pdf of the random variable pair (X, Y).
   (b) Write down the integral you would need to solve to obtain the marginal pdf f_Y of Y.
   (c) Find $\mathbb{E}(1/X)$. *Simplify any gamma functions.*
   (d) Find $\mathbb{E}Y$. *Hint: Use iterated expectation.*
   (e) Find $\mathbb{E}(1/X^2)$.
   (f) Find $\text{Var}\, Y$.
   (g) Find $\text{Cov}(X, Y)$.
   (h) Find $\text{corr}(X, Y)$.

4. Let X_1 and X_2 be independent random variables such that

$$X_1 \sim \text{Gamma}(\alpha_1, \beta)$$
$$X_2 \sim \text{Gamma}(\alpha_2, \beta)$$

   for some $\alpha_1, \alpha_2, \beta > 0$.
   (a) Find the mgf of the random variable $S = X_1 + X_2$.
   (b) Identify the distribution of S.
   (c) Find the mgf of

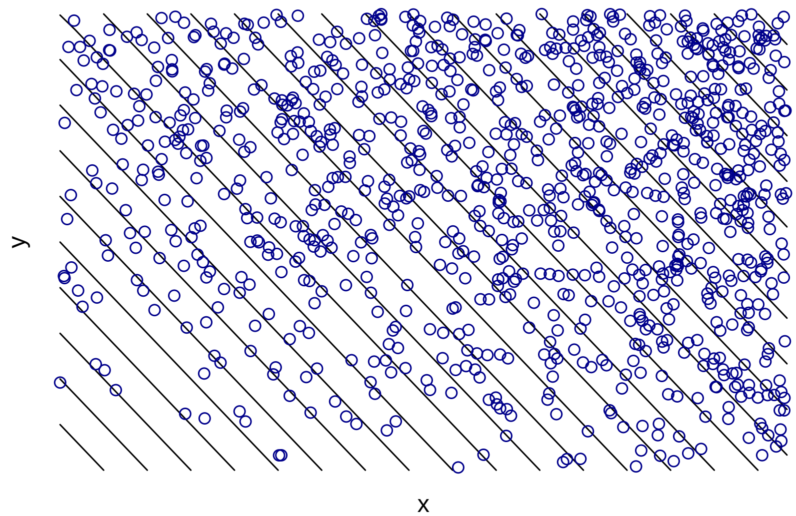
$$\bar{X} = \frac{X_1 + X_2}{2}.$$

   (d) Identify the distribution of $\bar{X}$.
   (e) Give $\text{Var}(X_1 - X_2)$.

5. Let the pair of random variables (X, Y) have the joint pdf given by

$$f(x, y) = \frac{2}{ab(a+b)}(x+y) \cdot \mathbf{1}(0 < x < a, 0 < y < b)$$

for some $a, b > 0$. The plot below shows contours of the joint pdf under some specific values of a and b with 1,000 realizations of (X, Y) plotted on top.



(a) State whether X and Y are independent and explain your answer.

(b) Find the marginal pdf f_X of X.

(c) Find the conditional pdf $f(y|x)$ of $Y|X = x$.

(d) Find $P(Y \leq b/2 | X = a/2)$.